Maths Beyond Limits 2023 Qualifying Quiz

We present to you five olympic problems and three exploratory ones. The problems are **not** sorted by difficulty. We will count only your **three** best grades from the olympic part – in particular, you don't need to solve everything, even if you want a maximum number of points. The exploratory problems are intended to be less standard and perhaps a little harder. They are worth 1.5x more points than olympic ones.

Do not get upset if you find the problems difficult as they are meant to be demanding and thought-provoking. Also, do not hesitate to send partial solutions, as they still might be awarded a significant number of points – especially in exploratory problems. If you have some interesting observations concerning introduced framework in the exploratory problems, or you find an intriguing question, share it with us in your solution!

You can use books or the Internet to look up definitions or formulas, but do not try to look for the problems themselves! In case the problem statement is unclear to you even after getting help from the aforementioned sources, please contact us (our email address is `mathsbeyondlimits@gmail.com`). You may not consult or get help from anyone else. Violation of any of these rules may permanently disqualify you from attending Maths Beyond Limits.

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1. Kuba's favourite number n has the following properties:

- sum of its digits is equal to the product of its digits,
- exactly half of its digits are ones,
- n is the biggest number with these two properties.

Determine n.

2. Krzysztof, continuing his career in forestry, got a tree – a connected graph with n vertices and no cycles. He turned each edge into an arrow, making it a directed graph, but he disliked the effect. Now he can choose any vertex v such that all edges incident to v are going in it, and he can reverse them, so that they go out of v. Prove that after some number of moves it is possible for him to obtain any orientation of edges he likes.

3. Let n be a positive integer. Zana and Marianna are playing a game. At the beginning, they have the set $S = \{1, 2, \ldots, n\}$. They take turns alternately. In one turn they choose an element $k \in S$, and then they remove all divisors of k from S. The player who can't make a move, loses. Zana plays the first move. Who has the winning strategy?

4. Let ABC be an acute triangle. H is its orthocenter, and O is its circumcenter. Let P be the midpoint of BH and Q be the midpoint of CH. Prove that the circumcircles of triangles PHQ and ABC are tangent if and only if $OH||BC$.

5. Let S be a union of polygons sitting inside three dimensional space. We know that the projection of S onto each of the three planes $x = 0$, $y = 0$ and $z = 0$ is a square with side length 1. What is the minimal possible area of S?

6. We call a positive integer n *lucky*, if we can fill a unit square with n rectangles (without overlapping), such that each of them has side ratio equal to $2:3$.

a) Determine all lucky numbers.

We call a positive integer n *serendipitous*, if we can fill a cube with n rectangular parallelepipeds (without overlapping), each of them with side ratio $1:2:3$.

b) Prove that each integer $n \geqslant 100$ is serendipitous.

c) What is the smallest possible C such that all integers $n \geqslant C$ are serendipitous? (Finding the exact answer here might be hard, so we expect some bounds like "C is greater than 15, but smaller than 73". The points will be dependent on how good is your answer in comparison to answers of other applicants.)

7. Let $d(X,Y)$ be the distance between two points in a plane. A function $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies $d(X,Y) = 1 \Rightarrow d(f(X), f(Y)) = 1$ for any points X, Y (in other words, if points X and Y are in distance 1 from each other, so are $f(X)$ and $f(Y)$).

a) Suppose f is injective, i.e. $f(X) \neq f(Y)$ if points X, Y are distinct. Does it imply that f is an isometry, i.e. $d(X,Y) = d(f(X), f(Y))$ for any X, Y?

b) What can you say when we don't require f to be injective?

8.

a) Let b_n be the number of labelled trees on n vertices (i.e. connected graphs without cycles, such that vertices are numbered from 1 to n). Prove that

$$b_n = \sum_{k=1}^{n-1} (n-k) \binom{n-2}{k-1} b_k b_{n-k}.$$

b) There are n witches sitting in a circle. Each witch has a hat with a number from 1 to n written on it (every number appears exactly once). In i-th minute (for $i = 1, 2, \ldots, n-1$) two witches swap places with each other. After $n-1$ minutes it turned out that for all $m = 1, 2, \ldots, n-1$ the witch with a number m is sitting on the place occupied at the start by the witch with a number $m+1$.

Let a_n be the number of possibilities in which this could happen. Prove that

$$a_n = \sum_{k=1}^{n-1} (n-k) \binom{n-2}{k-1} a_k a_{n-k}.$$

c) Deduce that $a_n = b_n$.

If you find any other interesting recurrence relations for sequences a_n and b_n, share them with us!