Maths Beyond Limits Balkans 2023 Qualifying Quiz

We ask you to solve **three** out of five *olympic* problems (i.e. problems 1-5) and **three** *exploratory* ones (i.e. problems 6-8). Do not get upset if you find the problems difficult as they are meant to be demanding, thought-provoking and get the best out of you. Also, do not hesitate to submit just partial solutions as sometimes they may be very near completion. This applies especially to problems 6-8, which are meant to help us understand your mathematical maturity.

If you have some interesting observations concerning the introduced framework in the exploratory problems, or you find an intriguing question, share it with us in your solution! These will also be awarded points even if you get stuck at the very beginning of the solution. Moreover, if you come up with some idea for a modified version of the presented problem, send it to us – this will be highly beneficial to your application!

You can use books or the Internet to look up definitions or formulas, but do not try to look for the problems themselves! In case the problem statement is unclear to you even after getting help from the aforementioned sources, please contact us. You may not consult or get help from anyone else. Violation of any of these rules may permanently disqualify you from attending any Maths Beyond Limits camp.

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1. Let M denote the set $\{1, 2, 3, ..., 2023\}$. A subset of M is called *Balkan* if the average of its elements is an integer. Prove that the number of all Balkan subsets is odd.

2. Triangle ABC is right-angled at point B and has incentre I. Points D, E and F are the points where the incircle of the triangle touches the sides BC, CA and AB respectively. Lines CI and EF intersect at point M. Lines DM and AB intersect at point N. Prove that $AN = BF$.

3. Determine all positive integers n such that $\lfloor n/k \rfloor$ divides n for all $1 \le k \le n$.

4. Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that

$$f(n + f(n))(m + f(n)) = (f(2m) + 2f(n))f(n)$$

holds for all positive integers m, n.

a) Find all possible values of $f(2022)$.

b) Find all possible values of $f(2023)$.

5. Let p be a prime and S a set of ordered pairs of remainders modulo p, such that:

(i) $(1, 1)$ is in S

(ii) if (a, b) is in S, then $(b, a + b)$ is in S

(iii) if (a, b) and (c, d) are in S, then (ac, bd) is in S.

Prove that all ordered pairs of remainders modulo p are in S.

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6. There is an equilateral triangle with side length n that is split into n^2 smaller equilateral triangles with side length 1, which we call unit triangles. Initially, one of the unit triangles is coloured black, the rest are purple. On each turn, we can pick one of the unit triangles and change its colour together with the colours of its neighbours (we call two unit triangles neighbours if they share a side). Our goal is to make the entire big triangle of one colour. Can we fulfil this goal if the unit triangle that is initially black is

a) any unit triangle not adjacent to a side of the big triangle?

b) one of three corner triangles?

c) one of the remaining unit triangles and n is even?

Can you say anything about the latter case when n is odd instead?

7. Let X be a set. We say that function $d : X \times X \to \mathbb{R}$ is a ***metric*** on X if it satisfies next 4 conditions:

(1) All $x, y \in X$ satisfy $d(x, y) \geq 0$ (*positivity*);

(2) $d(x, y) = 0 \iff x = y$ (*non-degeneracy*);

(3) All $x, y \in X$ satisfy $d(x, y) = d(y, x)$ (*symmetry*);

(4) All $x, y, z \in X$ satisfy $d(x, y) \leq d(x, z) + d(z, y)$ (*triangle inequality*).

Ordered pair (X, d) is called ***metric space***.
For any point x in metric space (X, d), and $r > 0$ we define ***open ball*** with radius r with

$$B(x, r) = \{y \in X \mid d(x, y) < r\}.$$

We define the distance **D(A,B)** between two sets A and B to be the minimum distance d between an element of A and an element from B, or the greatest number smaller than all distances between some element of A and some element of B if the minimum such distance does not exist.

To understand the concept better, prove that the following examples are metric spaces:

(1) On $X = \mathbb{R}$ and $d(x, y) = |x - y|$.

(2) On $X = \mathbb{R}^2$ and:

$$d_2((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$
$$d_1((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$$

Exercise 1: Let $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be a function defined with:

$$d((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\} + |x_1 - x_2|.$$

a) Prove that $(\mathbb{R}^2, d)$ is a metric space.

b) Describe the open balls $B((0, 0), 1)$ and $B((1, 3), 5)$.

Exercise 2: Let $d \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function defined with

$$d((x_1, y_1), (x_2, y_2)) = \begin{cases} |x_1| + |x_2| + 2|y_1 - y_2|, & y_1 \neq y_2 \\ |x_1 - x_2|, & y_1 = y_2. \end{cases}$$

a) Prove that d is a metric on $\mathbb{R}^2$.

b) Describe the open balls $B((0, 0), 1), B((1, 0), 1)$ and $B((2, 4), 5)$.

c) Let A and B be sets given with

$$A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 16\} \quad \text{and} \quad B = \{(x, y) \in \mathbb{R}^2 \mid y \geq 6 - x\}.$$

Find $D(A, B)$.

8. We are given n distinct positive integers $a_1, a_2, ..., a_n$. Our hypothesis is that among these n numbers one can always find two positive integers x and y such that $\frac{x}{gcd(x,y)} \geq n$. Prove that our hypothesis is true for

a) $n = p$, for p prime.

b) $n = p + 1$, for p prime.

c) $n = p + 2$, for p prime.

Can you find two n-tuples of distinct positive integers for every n that show that the right-hand side of the inequality from the hypothesis cannot be improved? Can you verify the hypothesis for any other cases of n?