Maths Beyond Limits 2022 Qualifying Quiz

As a part of the application process, we ask you to think about a subset of the following problems. Only the **five best solutions** will be taken into account – in particular, you don't need to worry about solving everything. Do not hesitate to submit partial solutions, as we are interested more in your way of thinking than in the result.

You can use books or the Internet to look up definitions or formulas, but do not try to look for the problems themselves! In case the problem statement is unclear to you even after getting help from the aforementioned sources, please contact us. You may not consult or get help from anyone else. Violation of any of these rules may permanently disqualify you from attending Maths Beyond Limits.

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1. Let $n \geqslant 2$ be a positive integer. The game starts with $2n$ piles of stones. Each pile has exactly one stone. Łukasz and Ania alternately merge two piles of their own choice, until there are only two piles left. Łukasz wins if those piles have even numbers of stones each, and Ania if the number of stones in each pile is odd. Łukasz starts the game. Who has a winning strategy, and what is it?

2. ABC is an acute triangle with $AB < AC$. Let O be its circumcenter, D be the foot of the altitude from A, M be the midpoint of BC and S be the projection of B onto AO. Prove that the circumcircle of the triangle DMS is tangent to the circle with diameter AC.

3. Simon painted each positive integer with some colour. It turned out that for any positive integers a, b numbers $a+b$ and a^2+b^2+1 have the same colour. Prove that actually all integers greater or equal to 2 have the same colour.

4. Triangle ABC satisfies the property that $\angle ABC = 2\angle ACB < 60°$. X and Y are the intersections of the perpendicular bisector of BC with a circle centered at A with radius AB. X is closer to BC than Y. Show that $3\angle XAC = \angle BAC$.

5. Is it possible to fill a $\mathbb{N} \times \mathbb{N}$ table (table with infinite length and height, but only in one direction) with all integers, without repetitions, such that each row is an increasing sequence and each column is a decreasing sequence?
See an explanatory figure for this problem at the bottom of the page.

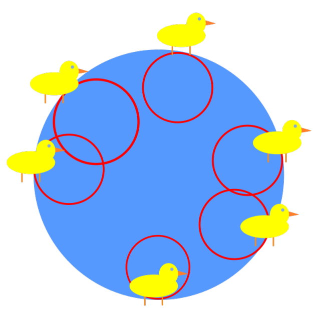
6. Let t be a positive integer. A sequence of positive integers $a_1, a_2, a_3, \ldots$ satisties $a_n = \gcd(a_{n+t}, a_{n+t+1})$ for any positive integer n. Does it follow that this sequence is constant?

7. An infinite flock of ducks sits on a coast of a circular lake. There is one duck per each point of the coast. Ducks are selfish, but not mean — every duck wants to have its own swimming area, in the shape of a disk tangent to the coast of the lake in the point it is sitting in, but they don't mind sharing that area with other ducks. Prove that there is a point on the lake that is in the swimming area of infinitely many ducks.

Remark. You may find the following fact useful: there is no sequence of points A_1, A_2, A_3, … containing all points of a circle.

-5	-3	3	
-4	0	4	
-1	2	5	

Problem 5: this 3x3 table satisfies the row and column conditions
(note that the table in Problem 5 is $\mathbb{N} \times \mathbb{N}$, not 3x3)



Problem 7: explanatory figure