Maths Beyond Limits 2021 Qualifying Quiz

We ask you to solve **three** out of five *olympic* problems (i.e. problems 1-5) and **three** *exploratory* ones (i.e. problems 6-8). Do not get upset if you find the problems difficult as they are meant to be demanding, thought-provoking and getting the best out of you. Also, do not hesitate to submit just partial solutions as sometimes they may be very near completion. This applies especially to problems 6-8, which are meant to help us understand your mathematical maturity.

If you have some interesting observations concerning introduced framework in the exploratory problems, or you find an intriguing question, share it with us in your solution! These also will be awarded points even if you get stuck at the very beginning of the solution. Moreover, if you come up with some idea to a modified version of the presented problem, send it to us — this will be highly beneficial to your application!

You can use books or the Internet to look up definitions or formulas, but do not try to look for the problems themselves! In case the problem statement is unclear to you even after getting help from the aforementioned sources, please contact us. You may not consult or get help from anyone else. Violation of any of these rules may permanently disqualify you from attending Maths Beyond Limits.

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1. Nonzero real numbers a, b, c satisfy the condition $a + b + c = -abc = -1$. Show that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq -1$.

2. Players $A_1, A_2, \ldots, A_{2020}$ competed in a chess tournament. Each player played exactly one match against every other player and there were no ties. We call a pair of players (A_i, A_j) *victorious* if there is no player A_k who won with both A_i and A_j. Determine whether it is possible that pairs $(A_1, A_2), (A_2, A_3), \ldots, (A_{2019}, A_{2020})$, (A_{2020}, A_1) are all victorious.

3. Positive integers $x,\ y,\ n$ satisfy $x \neq y$ and $x + y \mid x^{2n} + y^{2n}$. Prove that $(x-y)^{4n} > xy$.

4. A triangle ABC is given with $\sphericalangle BAC = 60°$. Let E and F be feet of angle bisectors from B and C respectively, and I be the intersection of BE and CF. M and N are midpoints of AE and AF, while P and Q are midpoints of IE and IF respectively. Prove that I lies on a line through circumcenters of triangles CMQ and BPN.

5. We call a positive rational number $\frac{p}{q}$ $(\gcd(p,q) = 1;\ p,\ q \neq 1)$ *balanced* if p and q are products of the same number of prime numbers (not necessarily distinct). Prove that for every positive integer k we may find two distinct positive integers x and y such that numbers

$$\frac{x+1}{y+1},\ \frac{x+2}{y+2},\ \ldots,\ \frac{x+k}{y+k}$$

are all balanced.

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6. Ania has a very large collection of finite 0-1 sequences.[1] In fact, her collection is infinite and she can't remember it. She wants to be able to quickly tell if any given 0-1 sequence is in her collection. She knows that for any sequence t in her collection, all subsequences of t also belong to it.

She wants you to find a finite set S of 0-1 sequences such that any 0-1 sequence is in Ania's collection if and only if it does not contain any sequence from S as a subsequence.

a) Can you help Ania no matter what is her collection of sequences?

b) If we would change all words "subsequence" to "substring", will the answer change?

Here we assume definitions of subsequence and substring as on Wikipedia.

[1] A 0-1 sequence is a sequence whose all elements are equal to either 0 or 1.

7. In a convex n-gon $A_1A_2\ldots A_n$ equalities

$$A_1A_3 = A_2A_4 = \ldots = A_{n-1}A_1 = A_nA_2 \quad \text{and} \quad A_1A_4 = A_2A_5 = \ldots = A_{n-1}A_2 = A_nA_3$$

hold.

For which n do they imply that this n-gon is regular? Investigate this question in cases

a) $n = 6$,

b) $n = 7$,

c) $n = 11$.

Any progress in case of general n will be also warmly welcomed.

8. Let n be a given positive integer. Justyna draws n distinct circles on a plane and colours every circle with a different colour. Then she draws a line l such that all circles are on one side of the line. Let P be an arbitrary point on l. If there exists a point A with colour c on some circle, such that $AP \perp l$ and segment AP does not contain points of any other colour, then paint P with colour c. After repeating such operation for every point on l, this line is split to some segments of various colours (some points of the line will remain uncoloured though).

a) What is the greatest number of these segments, in terms of n?

b) Tomek wanted to do the same thing as Justyna, however instead of circles he chose to use equilateral triangles. Moreover he placed his triangles and line l so that each triangle has a side parallel to l (i.e. all triangles are placed either as Δ, or as ∇). What will be the greatest possible number of segments this time?

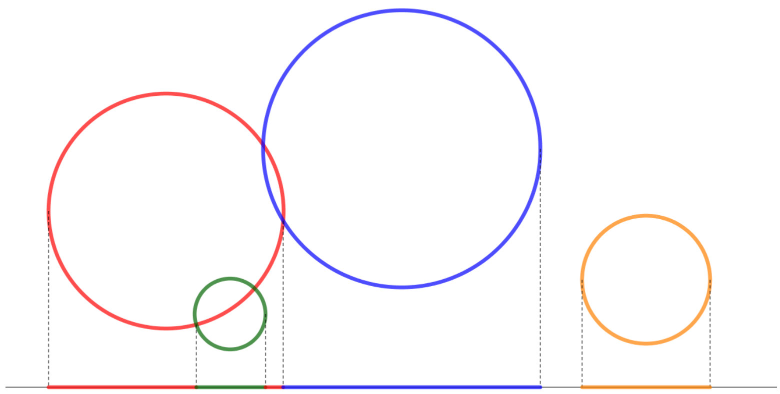

Figure 1: In this case we have five segments.