International Mathematical Camp

Maths Beyond Limits

Camp brochure

Milówka 2017

Visegrad Fund

Polish Children's Fund

Smart UP ADAMED

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About the camp

Maths Beyond Limits (MBL) is an international mathematical camp for talented youth. MBL 2017 was the second edition of this initiative and was held from 6th to 17th September 2017 in Milówka, Poland. 60 students from Czech Republic, Hungary, Poland and Slovakia (who filled in the **Applicant Questionnaire** and solved problems from the **Qualifying Quiz**), along with 11 tutors and 8 organisers, took part in it.

The aim of the camp is to create space for development of young maths enthusiasts through working on interesting and demanding mathematical subjects. It is designed to encourage participants to share their knowledge and passion with others as well as to enhance cooperation and integration of European mathematical societies. Moreover, our goal is to awaken youth's curiosity and to help them make important habits of creative thinking, self-development, ambition and ability to cooperate.

MBL 2017 was an 11-day-long programme, filled with numerous events. Every regular day there were three 80-minute-long blocks of **Mathematical Classes**. They were devoted to some of the most beautiful concepts in mathematics from outside the high school's curriculum. During every block there were 3 lecture sessions to choose from, each concerning different mathematical field, which contributed to the diversity of academic experience participants had on the camp. Classes were followed by 45-minute-long **TAU**, which stands for **Time Academic Unscheduled** that was designated for the students to work on the problems independently or with the help of tutors. It was also a great opportunity to clarify any points of the classes that were found hard or insufficiently explained. The last regular mathematical events were **Camper Talks**, i.e. 30-minute-long presentations given by some participants on topics connected to mathematics they were interested in. Their main goal was to give participants the opportunity to practice important skills of clear presentation of mathematical topics in English while inspiring others with their passions. Students had prepared them beforehand and consulted them with staff members during the camp.

Maths was not the only thing the camp was about, therefore the hard work was followed by **Evening Activities**. They were fun, challenging, educational or thought-provoking. Most of them were run by the participants, while other by the staff. Lastly, there was also time for **Sports** in the beautiful scenes of Beskidy mountains. Activities included running as well as playing volleyball, football and frisbee.

On MBL we do not to let anyone get bored, therefore also quite a few special events were organised. We went for a **Hike** in Beskidy mountains and solved a mystery during the **Night Criminal Game**, as well as had fun playing guitars and singing by the **Campfire**. Participants also got a chance to talk about careers, universities and olympiads with tutors and organisers during the **Questions Evening Café**. Moreover, they were able to improve their problem solving skills thanks to **Relays** (team competition similar to Náboj), **Mathematical Matches** and **Olympic Challenge**. There were 3 divisions of Mathematical Matches differing in difficulty, each of them was a team contest where teams got problem sets in the evening and presented their solutions by the board the next day. If students wanted to train a specific branch of mathematics, they were able to do so in the Olympic Challenge. If they solved a given problem, they were rewarded with a next one and if they got stuck, they could ask staff members for hints. All these little things contributed to making MBL 2017 such a great, inspiring and unforgettable camp.

Sponsors and project partners

Along with numerous people that devoted their free time to come to the camp and help in situ stand the organisations that assisted the Organising Committee in financial and administrative way. Making MBL 2017 would not be possible if it were not for the generous sponsors and wonderful project partners, which we had great pleasure of cooperating with.

Sponsors

The International Visegrad Fund
visegradfund.org

ADAMED SmartUP
adamed.com.pl

Project partners

Polish Children's Fund
fundusz.org

Charles University
mff.cuni.cz

The Joy of Thinking Foundation
agondolkodasorome.hu

Trojsten
trojsten.sk

People of the camp

The following is the list of all people that took part in MBL 2017 together with the titles of the activities they ran or functions they had during the camp. **MC** stands for Mathematical Class, **CT** for Camper Talk, **EA** for Evening Activity. **NCG** means that the person was one of the actors in Night Criminal Game.

Participants

- Angelika Andó
- Zsuzsanna Baran
- Jagoda Bracha
  EA: Polish song
- Balázs Bukva
- Filip Čermák
- Tímea Csahók
  EA: Intro to Latin
- Jakub Dobrowolski
  CT: Every closed loop has an inscribed rectangle
- Krzysztof Druciarek
- Witold Drzewakowski
- Mikołaj Grzebieluch
- Soma Hansel
- Veronika Hladíková
- Pavel Hudec
- András Imolay
- Orsolya Lili Janzer
- Jakub Kamiński
  CT: Pell's equation
- Mikołaj Kamiński
- Viktória Klász
- Lenka Kopfová
- Aleksandra Kowalska
  CT: Non-euclidean geometries
- Lucia Krajčoviechová
- Natalia Kucharczuk
  EA: Pique turns
  EA: Hot chilli pappers sauces making
  NCG: Slavia
- Weronika Lorenczyk
- Tomasz Makowski
  EA: Rubic cube intro
- Mateusz Masłowski
- Dávid Nagy
- Damian Obara
  EA: Improvisation workshops
  EA: Spontaneous movement
  EA: Intro to Swedish cinematography
- Radek Olšák
- Mikołaj Pacek
  EA: "What song is it?"
- Iwo Pilecki-Silva
  NCG: Environ
- Olaf Puchalski
- Ákos Záhorský
- Alicja Ziarko
  NCG: Hag
- Martin Zimen
- Kristóf Zólomy
- Radosław Żak
- Michał Pychtin
  EA: Vast range of quizzes
  EA: Number quiz
- Michał Radwański
  CT: Coq - automatic theorem prover
  EA: Making kebap
- Hedvika Ranošová
  EA: Singing workshop
- Filip Rękawek
- Piotr Ryłko
- Mateusz Rzepecki
- Michał Siennicki
- Philip Smoleński
  EA: Vast range of quizzes
  EA: Number quiz
- Katarína Studeničová
- Jakub Suchánek
  CT: Random walks on graphs
- Zuzana Urbanová
- Miléna Vankó
- Vince Velkey
  EA: Intro to Latin
- Kacper Walentynowicz
  EA: Contract bridge workshops
- Jakub Wornbard
  EA: Contract bridge workshops

Semitutors

- Radomił Baran
  MC: Projective geometry
- Maciej Dziuba
  MC: Power of point and radical axes
- Jakub Grudzień
  MC: Symmedians
- Jan Kociniak
  MC: Modular arithmetic
- Mieszko Komisarczyk
  MC: Projective geometry
- Paweł Poczobut
  MC: Commutative algebra
  EA: Vast range of quizzes
  EA: Number quiz
- Tomasz Przybyłowski
  MC: Cloudy continuum hypothesis
  NCG: Woodman
- Mariusz Trela
  MC: Extensions of $\mathbb{F}_p$
- Kada Williams
  MC: p-adic numbers
  MC: Symmetric polynomials

Tutors

- Łukasz Bożyk
  MC: Feel the flow
- Robert Crumplin
  MC: Counterexamples in analysis
  MC: Countability
  MC: Topics in group theory
- Gábor Damásdi
  MC: Finite geometries
  MC: Combinatorial geometry
- Ján Gašper
  MC: Logic
- Andrzej Grzesik
  MC: From tic-tac-toe to Ramsey theory
- Marcin Michorzewski
  MC: Compass and streightedge - constructions
  MC: How to win elections
- Leszek Sołdan
  MC: Probabilistic method
- Adela Kostelecka
  MC: Introduction to group theory
- Marian Poljak
  MC: Academic number theory
- Štěpán Šimsa
  MC: Complexity
- Radovan Švarc
  MC: Introduction to topology
  MC: Connectedness and sphere eversion

Organisers

- Piotr Ambroszczyk
  Math. Matches director
  MC: Projective geometry
- Dominik Cieśliński
  Purchasing manager
  EA: Power-Point karaoke
  EA: Crêpes artistiques
  NCG: Drunkie
- Grzegorz Dłużewski
  Relays director
- Mateusz Kobak
  MCs director
  MC: Introduction to Galois theory
- Ania Łeń
  Financial coordinator
  MC: How to win elections
- Marta Mościcka
  Website administrator
- Paweł Piwek
  EAs coordinator
  Media coordinator
  NCG director
  MC: Topics in group theory
  EA: Cardboard throne
- Szymon Zwara
  Sports coordinator
  Promotion director

Others

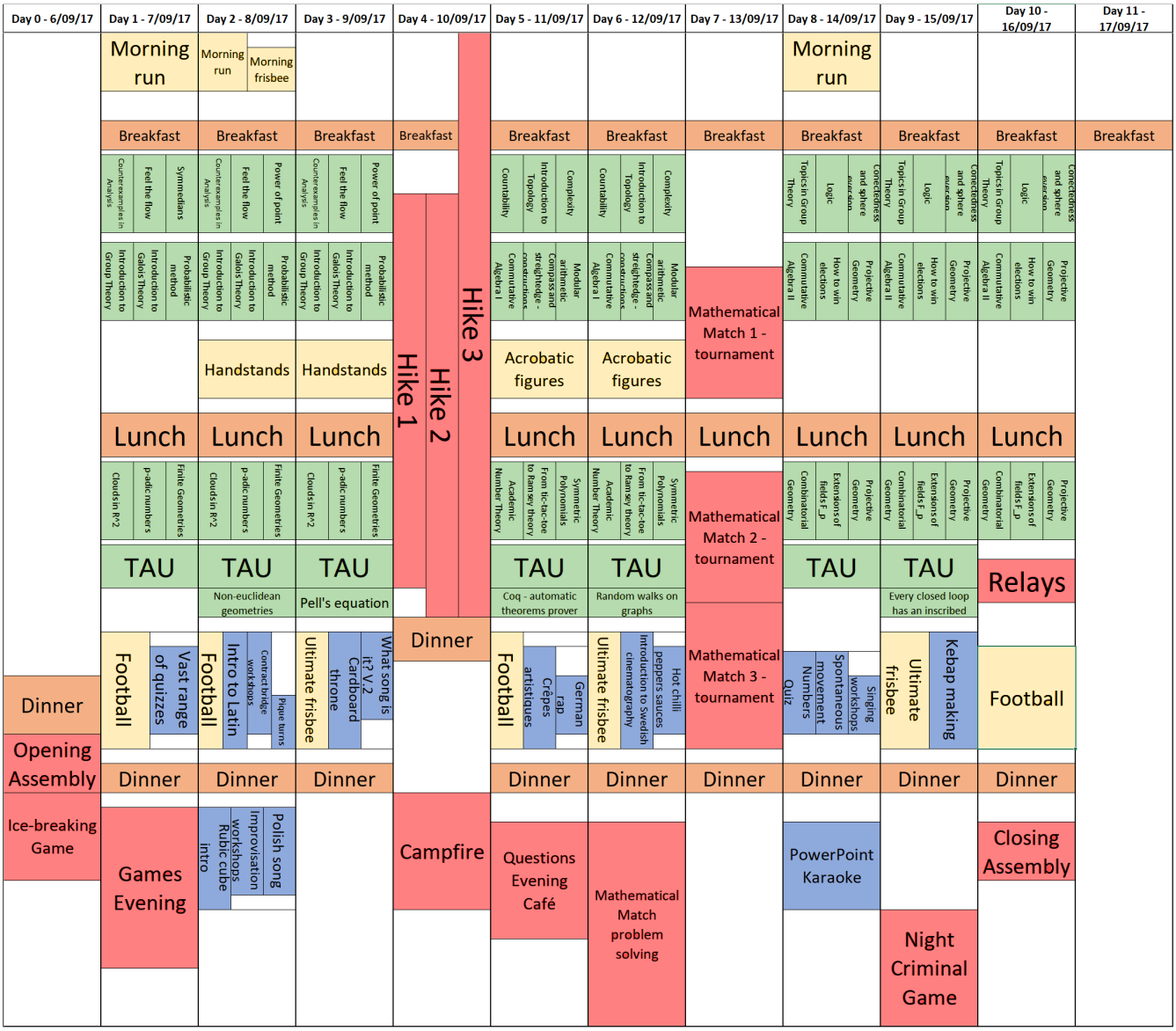
- Sebastian Gołojuch
  Camera operator
  Film editor
- Michał Wiraszka
  Photographer

Testimonials

- **Jagoda Bracha (participant)**: For me the best in MBL was: making new friends and an atmosphere encouraging socializing with other people, opportunity to talk to people from abroad and practice English, variety of non-mathematical activities and good quality of mathematical classes, which aren't only olympic (which is a big problem at most of camps). I was trying to use the camp maximally, so I didn't have any time to just sit in my room. Also is the only camp I've been to where organizers and participants are equal, nobody is forced to do something (for example attend classes or go to sleep).
- **Witold Drzewakowski (participant)**: It was my first experience of this kind. The lectures I attended made me want to know more and to learn more. This was an opportunity to meet people from different countries, have fun, play, hike, talk & study. I appreciate your hard work guys you put in organizing MBL.
- **Mikołaj Kamiński (participant)**: Maths Beyond Limits is the best mathematical camp I've ever been to. It is so great because of so many friendly people that make this wonderful atmosphere here. Every day of the camp is unique thanks to many diverse lectures to attend or activities to participate in. I highly recommend it to anyone who is interested in exploring maths or making new friends.
- **Viktória Klász (participant)**: MBL has been a very special experience for me. It is most probably the only camp where you can take part in a real mathematical match, sing Polish songs, investigate a crime in the night criminal game and get a cool hoodie on top of that. If you would like to attend interesting mathematical classes and meet amazing people who are also interested in maths, Maths Beyond Limits is the best place for you!
- **Vince Velkey (participant)**: Amazing 11 days. I learnt a great deal during the lectures and could get to know smart fellows from the V4.
- **Kacper Walentynowicz (participant)**: MBL was just amazing. It was a great pleasure for me to participate in the camp along with all those awesome people. Never have I experienced such a friendly atmosphere on a science camp.
- **Jakub Wornbard (participant)**: The camp was incredible. I had fun in every second of it and still managed to learn a lot. A great mix of interesting people, challenging lectures and unusual activities.
- **Maciej Dziuba (semitutor)**: MBL was one of the best camps in my life. I met a lot of great people, spent my time with them and obviously I learnt maths.
- **Kada Williams (semitutor)**: I loved meeting so many different people: it was brilliant to see everyone's approach to maths and what made them passionate about it.
- **Robert Crumplin (tutor)**: It was a great opportunity to work with talented individuals and do some interesting maths, all while experiencing Poland; something I will never forget!

Camp Schedule

The camp is packed with events so that nobody is bored at any time. A wide range of lectures, talks, problem sessions and other activities provides opportunities for skills and knowledge development regardless of participants' specific interests. They can benefit from it both if they are keen on academics or if their cup of tea is contest maths. Details (and/or invitations) of all activities organised at MBL 2017 are given on the following pages.



Mathematical Classes

- **Counterexamples in Analysis - Robert Crumplin**
  Exploring some interesting and maybe counterintuitive problems associated to key results in real analysis. We will go over some standard theorems on sequences and series, continuity and differentiability (1 per class) and look at related problems.

- **Feel the flow - Łukasz Bożyk**
  Flows in graphs constitute a beautiful and powerful part of graph theory. Imposing a structure of a flow network on a graph allows for drawing unexpected conclusions, sometimes deriving elegant proofs of classical theorems, e.g. Hall's marriage theorem. The subject is fruitful in both classical (Ford & Fulkerson, 1956) The course will consist of three parts. The first one will be devoted to basic definitions, concepts and propositions. The second one will concern the main theorem of Ford and Fulkerson and its applications. The final part will focus on a fresh approach of Marks and Unger, and its potential to prove some old (and hopefully some new!) combinatorial theorems.and modern (Marks & Unger, XII 2016) results. No prerequisite knowledge is needed but some experience with graphs will be welcomed.

- **Power of point and Symmedians - Maciej Dziuba, Jakub Grudzień**
  Olympic problems with the use of Radical Axis, Power of Point andw Symmedians.

- **Introduction to Group Theory - Adéla Kostelecká**
  Introduction to Group Theory. No prerequisites.

- **Introduction to Galois Theory - Mateusz Kobak**
  The class will intoduce the idea of field extensions leading to basis of Galois theory. We will follow to Solvability Theorem, as well as other uses of the theory we build. Knowledge of basic group theory is required, elementary ring theory and linear algebra is recommended.

- **Probabilistic method - Leszek Sołdan**
  Probabilistic method is a powerful and very elegant tool in tackling olympiad and research problems from various areas, such as analysis, game theory, graph theory, geometry and number theory. The basic idea is really simple: In order to prove the existence of a structure with certain properties, we construct an appropriate probability space and show that a randomly chosen element in this space has the desired properties with positive probability.
  (a) Prove that among 650 points within a disc of radius 16 we can find 10 that can be covered with a ring with inner and outer radius equal to 2 and 3 respectively.
  (b) Prove that any set A of natural numbers contains a subset B such that the equation $a + b = c$ has no solutions in B and $|B| \geq |A|/3$.
  (c) Determine the maximal number of subsets of the set $\{1, 2, ..., n\}$ such that none of them is contained in another one.

  If you can solve all the above problems and don't like candies, then go play frisbee. All the others are highly encouraged to come!

- **Clouds in $\mathbb{R}^2$ - Tomasz Przybyłowski**
  We will show how covering the plane with clouds is connected to Continuum Hypothesis. A class for those who like Set Theory and Logic.

- **p-adic numbers - Kada Williams**
  Imagine that we could talk about infinite sums like $a(0) + a(1)p + a(2)p^2 + \ldots$, where the $a(i)$'s are from $\{0, 1, ..., p-1\}$. Just like on real numbers (writable in base p), we can create analysis in this "base $1/p$" number system. However, our notion of distance involves thinking mod p^k, which makes p-adic analysis often quite bizarre. We explore topics such as p-adic logarithms and Taylor series, and more! Knowledge of real analysis is recommended but not required.

- **Finite Geometries - Gábor Damásdi**
  Finite geometries are highly symmetrical mathematical structures. They have similar properties to the Euclidean plane but they have finitely many points and lines. These structures can be found in many areas, such as coding theory or competition scheduling. There are also some interesting card games that are based on these structures. In this class we will work through some problems that will help us understand the basic definitions and examples. The class has no prerequisites.

- **Countability - Robert Crumplin**
  Introduction to set countability and cardinality. We will prove standard results about the cardinalities of familiar sets such as the naturals, rationals, reals, ... and use these to investigate some interesting problems.

- **Introduction to Topology - Radovan Švarc**
  We will be concerned with metric spaces and basic topology. No previous knowledge necessary.

- **Modern topics in Graph Theory - Štěpán Šimsa**
  We will cover different topics in Graph Theory and discover modern branches of this field such as complexity theory.

- **Commutative Algebra I - Paweł Poczobut**
  Introduction to rings and modules. Very basic Group Theory required.

- **Compass and streightedge - constructions - Marcin Michorzewski**
  Probably you have solved some geometric problems. Thus, you have been probably drawing many lines and circles. But do you know what exactly can be done using only a compass and a ruler? If you want to discover the answer using Galois Theorem, algebra and polynomial equations and also learn about Steiner-Poncelet theorem to figure out whether it is possible to construct every Euclidean Construction just with a straightedge, then you should come to this class!

- **Modular arithmetic - Jan Kociniak**
  The lecture is designed for olympiad contestants who want to train their skills in modular arithmetic (especially modulo primes). The basic and more advanced theorems will be shown (i. e. Euler's theorem, Thue's lemma, bunch of facts about order of an element modulo prime, primitive roots and quadratic residues) and we will focus on solving problems taken from national and international olympiads or other sources. It doesn't matter if you are a beginner or an advanced problem-solver, don't hesitate and come!

- **Academic Number Theory - Marian Poljak**
  Dirichlet's theorem and it's augmented versions, Farey fractions and Hurwitz theorem, pehaps Liouville's theorem or Pell's equation. second lecture - definitely the 4-square theorem with an elementary proof, then perhaps some more academic stuff - lattices, Minkowski general theorem and another (geometric) proof of the 4-square theorem. If there will be time, I'd like to add quadratic reciprocity as it has olympiad uses. My main source is the university Number theory course I took this year.

- **From tic-tac-toe to Ramsey theory. - Dr Andrzej Grzesik**
  During the classes we will consider some generalizations of the tic-tac-toe game and try to find optimal strategies in them, or at least prove their existence. This will lead us to certain important results and open problems in Ramsey theory.

- **Symmetric Polynomials - Kada Williams**
  This class is intended for beginners who would like to deepen their mathematical perspective on symmetric polynomials and how they allow us to prove properties of certain real numbers (e.g. $\sqrt{2}+1$) using their "conjugates". As a finale, we prove that the sum of square roots of some positive integers is equal to 0 only in trivial cases.

- **Topics in Group Theory - Robert Crumplin, Paweł Piwek**
  A look at the structure of finite groups. Namely simple groups, nilpotent groups, automorphisms and related theorems such as the Sylow theorems, Jordan-Holder theorem and finite group classifications.

- **Logic - Ján Gašper**
  Logic values; AND, OR, IF, IFF, NOR, XOR, and how to work with them; binary representation of natural numbers; basic operations in binary (adding, multiplying); problem of adding two natural numbers in binary; problem of adding two natural numbers in terms of logic and logic gates; modelling "adding circuit" for 1bit integer; modelling "adding circuit" for 8bit integer (and n-bit integer); integer overflow and negative numbers; problem of multiplying two integers; (if there is time left:) basic logic gates from transistors.

- **Conectedness and sphere eversion - Radovan Švarc**
  We will be concerned with compactness and connectedness in topology. Basic topology knowledge required. We'll take a look at how to turn a sphere made from self-passing material inside out without cutting or creasing it.

- **Commutative Algebra II - Paweł Poczobut**
  Continuation of Commutative Algebra I. We will cover advanced rings' and modules' properties, as well as Nullstellensatz which is a fundament of algebraic geometry.

- **How to win elections - Anna Łeń, Marcin Michorzewski**
  Election seems to be straightforward - you just go and vote. But is the maths and idea behind it also so straightforward? If you want to see why it really isn't and also why introducing single-seat regions may be dangerous, why apportionment is not always fair and why voting systems are never ideal, it's a perfect class for you. Covered topics will include Arrow's impossibility theorem, Concordet's paradox, gerrymandaring, apportionment and possibly some more. You'll spend some part of the classes solving (fun) problems and looking at the real election data!

- **Projective Geometry - Piotr Ambroszczyk, Radomił Baran, Mieszko Komisarczyk**
  During classes we will start with projective and affine transformation of a line. We will introduce definition of projectivity, perspectivity and Steiner conics. After that we move on to plane transformations. Starting from basic definitions and prove of the well-known properties we will end with solving quite advanced problems. There will be shown prove that we can find unique affine/projective transformation mapping 3/4 points onto another 3/4 points respectively. The last part of the classes will be focused on conic's section. First of all, some basic lemma and conics' properties will be presented there. Afterwards there will be proven that isogonal/isotomic conjugate of a line with respect to given triangle is some conic. Finally we will introduce different kinds of hyperbola – Feuerbach, Kiepert and Jerabek hyperbolas. For all presented theorems and lemmas there will be presented examples and usage in olympic problems.

- **Combinatorial Geometry - Gábor Damásdi**
  Combinatorial geometry is a mixture of interesting mathematical areas such as geometry, topology, combinatorics and algebra. Most of the problems are easy to understand but the solutions require some clever ideas. For example, a typical question would look like this. Can you always cut a cake into six parts, such that each part has the same amount of candles on it? What if you can only use your knife three times? We will work through some problems to understand the basic definitions and ideas, and I will also talk about the active research areas in this eld. The class requires just the basic understanding of euclidean geometry.

- **Extensions of fields $\mathbb{F}_p$ - Mariusz Trela**
  Do you ever feel restricted in $\mathbb{Z}_p$, as if it was missing something? We'll talk about how to add "nonexistent" elements in a way that makes sense and what we gain from it. In the end we'll see how we can prove the law of quadratic reciprocity using field extensions, a proof less known and perhaps more intuitive than the usually shown Eisenstein's proof. Basic knowledge about modular arithmetic required.

Camper Talks

- **Non-euclidean geometries - Ola Kowalska**
  Talk supervisor: Mateusz Kobak

  In the talk, first some basic properties of spherical geometry are shown and then some of its interesting theorems. At the end, we prove Euler's formula for convex polyhedra, which uses spherical geometry.

- **Pell's Equation - Kuba Kamiński**
  Talk supervisor: Paweł Piwek

  This talk provides basic information about the Pell's Equation. The proof and consequences are explained, and some examples of problems solvable using this theorem are provided. We also include a few small, but useful facts from Number Theory.

- **Coq - automatic theorem prover - Michał Radwański**
  Talk supervisor: Grzegorz Dłużewski

  Coq, automatic theorem prover is a computer system based on algorithmic approach to solving problems from the axioms, which relies on Curry–Howard isomorphism, a correspondence between mathematical proofs and computer programs. In the talk its capabilities are briefly presented.

- **Random walks on graphs - Jakub Suchánek**
  Talk supervisor: Szymon Zwara

  In the talk we try to calculate average time it would take to walk (randomly) from given vertex A to another given vertex B on various graphs and then derive the general formula.

- **Every closed loop has an inscribed rectangle - Kuba Dobrowolski**
  Talk supervisor: Grzegorz Dłużewski

  We prove that on every closed loop there exist four points that make a rectangle. To unravel the mystery a little bit, we start with defining a function that takes two points A and B on the loop and puts another point over the midpoint of the segment AB at the altitude of the length of the segment AB. We prove this isn't injective using topological methods.

Special Mathematical Events

- **Relays**
  Relays was a mathematical team contest with team consisting of 5 campers. At each moment of this hour-long contest there were exactly 6 problems on the table of each team. When someone was sure of their answer to one of the given problems, they passed the solution to the person of the staff assigned to them. And although it may seem really similar to Náboj, there is one quite a significant difference. Namely, once in a while one could stumble upon an unusual problem - to be precise, a game. To solve it contestants needed to go to Grzegorz and play with him the game as specified in the problem statement. They were given the choice of who goes first and they gained a point if and only if they won the game. Unlike in other type of problems, each team had only one chance to play each game.

- **Mathematical Matches**
  Some participants may have fancied some more challenging problems than those on Relays and Mathematical Matches were perfect for this purpose. In each Mathematical Match, two teams of 10 people were given 11 advanced mathematical problems to be solved by them within 20 hours. However, the aim was not only to solve them but also to explain them to everyone in the team, since after these 20 hours elapsed, there was a tournament, in which (not delving much into details) everyone could be chosen to present solution to any of the problems by the board.

- **Olympic Challenge**
  Although there was an abundance of Mathematical Classes, Camper Talks, Relays and Mathematical Matches in the offer, one may have felt that they cannot develop their problem-solving skills in a specific field as much as they would like to. Indeed, due to the vast range of personal preferences as well as the variety of mathematical levels among the participants, it seemed unfeasible to cater needs of everyone. But was it in fact?

  The solution to this demand was Olympic Challenge with two divisions - Younger and Older - each of them consisting of 9 problems in each of the following branches of olympic mathematics: algebra, combinatorics, geometry and number theory. First problems from each branch and each division have been published at the beginning of the camp. From this moment, everyone could approach to solve them. Once they have solved them, they would go to the appropriate staff member and explain him the solution. If the solutions were correct, they were given the next problem in the same branch of the same division. However, it was possible that they would not be able to overcome the problem by themselves, hence to avoid the situation of them struggling against one problem for the whole camp, they could come to that appropriate staff member, discuss the approaches they had taken to tackle this problem and/or get some hints.

Evening Activities

- **Vast range of quizes - Paweł Poczobut, Michał Pychtin, Philip Smolenski**
  It's time for a quiz! If you enjoy trivia, especially those on random facts, it's definitely something for you, but we'll try to have some good laughs, too. If all that is still not enough - come for cookies, coffee and tea! Fun guaranteed.
- **Contract bridge workshops - Kacper Walentynowicz, Jakub Wornbard**
  During our Evening Activity we would like to acknowledge people with contract bridge. Bridge is a card game that is no longer just a typical card game for groups of four people, but a worldwide recognizable sport, like chess.
- **Intro to Latin - Tímea Csahók, Vince Velkey**
  The aim of our Evening Activity is to introduce the attendants to the basics of the Latin language while having fun. In the first part, we'll discuss some words of the English language that have Latin origin – to prove how much effect Latin has on modern Languages. The next step will be a general review about the grammar of Latin, then we'll dive into the abyss of nominal and verbal inflection. Then we will practice on grammar tests and easy texts. Finally we also want to introduce some of the most famous antic authors.
- **Pique turns - Natalia Kucharczuk**
  Firstly, I'll show what ballet positions look like. It'll take some time but after that we'll go on to the main part of this Evening Activity - we'll start doing turns! I'll teach participants step by step how to do chaines turns. It's really easy and it's a good start for everyone - no matter how skilled someone is. Afterwards, the campers will finally learn pique turns (naturally, also step-by-step). I hope it'll be a fun and lots of people will enjoy spinning randomly around with me.
- **Rubic cube introduction - Tomasz Makowski**
  During this EA you can learn how to solve Rubic cubes, not only the most popular 3x3 type.
- **Improvisation workshops - Damian Obara**
  Great fun, integration, focus and sense of body exercises.
- **Polish song - Jagoda Bracha**
  During my Evening Activity, I will teach participants to sing two traditional songs from Lubelszczyzna, where I come from. Making music together was a popular form of spending time some time ago and I believe that everyone actually can sing. I think that it can be an interesting experience, good fun and a linguistic challenge for non-Poles (and for Poles also a bit).
- **Cardboard throne - Paweł Piwek**
  Paweł's exuberant love for cardboard has become quite significant during the first MBL. This time we want to make something huge. If you ever wondered how it is to be an architect or a decorator - do come!
- **What song is it? V.2 - Mikołaj Pacek**
  "What song is it?" is an interactive quiz show based on famous polish show "Jaka to melodia?". We will divide us into three teams and test our knowledge of the biggest musical classics. The winning team will get a special prize. If you are a fan of music and want to compete with other people then come and join us.
- **Crêpes artistiques - Dominik Cieśliński**
  The activity is about making pancakes, which not only taste great, but also look so. We'll try to make them into paintings or brand them with MBL logo.
- **German rap - Mieszko Komisarczyk**
  During my EA I would like to present the most influential and important German rappers since 2000. Watching the interviews and listening to certain songs will be a mainpoint of performance.
- **Intro to Swedish cinematography - Damian Obara**
  We will see into works of Igmar Bergman, one of the greatest directors of 20th century. We will watch his film and have a discussion.
- **Making sauces... with hot chilli peppers! - Natalia Kucharczuk**
  During this Evening Activity, we'll make hot sauces and eat them with tortilla chips. We will use some peppers like jalapeno or chilli as well as pineapples, tomatoes and garlic - it all depends on which ingredients will be available. It may be really hot there!

- **Number quiz - Paweł Poczobut, Michał Pychtin, Philip Smolenski**
  Have you ever wondered how many grains of rice an average Chinese eats every day? How many A4 pages can you fill black with single BIC pen ink? Or maybe how many toothbrushes are sold in Texas every year? If you enjoy such questions, we invite you to take part in our quiz! No knowledge required. Lots of fun guaranteed.

- **Singing workshops - Hedvika Ranošová**
  At this activity, we'll try to learn a few polyphonic songs. You don't have to be perfect singers - the chosen songs should be quite easy. If anybody wouldn't want to sing, but rather help us with a musical accompaniment (playing for example keyboard or flute), it would be great, too.

- **Spontaneous movement - Damian Obara**
  If you like to dance but feel embarrassed or afraid what others would say, this is something for you. Some psychedelic music and untrammelled movement.

- **PowerPoint Karaoke - Dominik Cieśliński**
  The activity is all about improvisation and has some stand-up in itself. In short - you present a PowerPoint presentation, which neither you nor the audience has seen before. Usually it is made random enough so that you need much creativity to give it some sense - which is where the fun comes from. It is an enjoyable and memorable experience.

- **Kebap making - Michał Radwański**
  I will show how to make your own kebap. We'll see the exact process of getting this delicious food on our plates. This will include forming the meat mass, getting it to the form most of us know under name "kebap", then cutting it off, spicing and most importantly, eating!

Special Events

- **Opening Assembly**
  An opening meeting were the staff introduce themselves, talk about what would be happening at the camp, explain the camp rules and officially open it.

- **Ice-breaking game**
  A competitive team game, whose goal is to induce integration and cooperation between participants that didn't know each other before the camp. Also, it is supposed to get people accustomed with the accommodation premises. The teams go to staff persons to recieve cooperative tasks and get points for them.

- **Games evening**
  An opportunity to chill out with new-made friends over some board and card games or playing mafia. Much of Set included!

- **Hikes**
  Taking place in the beautiful mountain scenery of Beskid Żywiecki, the camp is a perfect opportunity to enjoy some hiking! There were three hiking routes differed by length and difficulty. Mellow Hike starts with taking a train to Rajcza, then climbing to Hala Boracza and going down to Milówka. Middle Hike takes the black route to get to Barania Góra, go down through Schronisko Przysłop to yellow route and then down to Milówka. Hardcore Hike starts with going by bus to Węgierska Górka, then taking the red route to Schronisko Rysianka, following the green one to Hala Boracza and joing the Mellow Hike on the green route down to Milówka.

- **Campfire**
  After the hikes there is a campfire with sausages, guitar playing and singing. "Let it go!", "House of the Rising Sun", "Coconut song", "What does the fox say?" and "Numb" are musts, but the variety of different music styles gives everybody a little bit of what they like.

- **Questions Evening Café**
  Chill atmosphere, candlelight, tea and biscuits make a great chance to talk to tutors and camp staff about choosing university, mathematical career, organising youth camps, training for olympiads and about other mathematical programmes.

- **Night Criminal Game**
  A real time life role play, where the participants create teams of police investigators trying to understand characters motives, make out who of them are lying, reconstruct the whole story and find out who was the murderer! Preceded by a movie-like trailer.

- **Closing Assembly**
  A time-to-say-goodbye camp closing meeting being a chance to thank everybody for participating and take some great group photos.

Sports

There are plenty of good reasons to be physically active. Physical exercise, especially the aerobic one, have positive effects on brain function on multiple fronts, ranging from the molecular to behavioral level.

Exercise helps memory and thinking through both direct and indirect means. The benefits of exercise come directly from its ability to reduce insulin resistance, reduce inflammation, and stimulate the release of growth factors - chemicals in the brain that affect the health of brain cells, the growth of new blood vessels in the brain, and even the abundance and survival of new brain cells. Indirectly, exercise improves mood and sleep, and reduces stress and anxiety. Problems in these areas frequently cause or contribute to cognitive impairment. (Source)

This is why we let MBL participants do all kinds of sports which can be organised. Variety of activities allows them to choose the ones which they enjoy and may benefit from the most.

- **Morning Run**
  Everything takes on breathtaking beauty in the warm hues surrounding a sunrise, especially something as magnificent as the scenery of Beskidy. Its intrinsic beauty and clean fresh air definitely compensate for waking up early. Having a physical activity in the morning, which does not collide with any other scheduled events, gives runners the energy boost to be wide awake during Mathematical Classes and other activities.

- **Football, Ultimate Frisbee**
  Other people might feel that playing team games is something they enjoy more than running so to meet their expectations we organise also games of football and ultimate (frisbee) on 30m by 62m artificial grass pitch. Although team sports rely on rivalry, we make sure to keep a friendly and welcoming atmosphere during games and everyone can join the game, regardless of their skill.

- **Volleyball, Frisbee, Acrobatic Figures, Handstands and more**
  The whereabouts of the “U Kubiców” guesthouse in Milówka offer a lot of space for outdoor activities such as frisbee, acrobatic figures or handstands. There is also a place with a volleyball net. Campers use this space in their spare time by playing and having fun together.

Handouts

Jan Kociniak - Number Theory

1 Theory

As the number theory is focused mostly on integers, we consider all of the numbers mentioned in the following theorems being integers, unless stated otherwise.

1.1 Basic theorems

Theorem 1.1.1 Let a, m be integers and $m > 0$. A number $\widetilde{a}$ for which the congruence $a\widetilde{a} \equiv 1 \pmod{m}$ is true is called **the inverse** of a modulo m. There is only one inverse of an element modulo m (in other words, there are no numbers $\widetilde{a_1}$ and $\widetilde{a_2}$ such that $\widetilde{a_1} \not\equiv \widetilde{a_2} \pmod{m}$ and $a\widetilde{a_1} \equiv a\widetilde{a_2} \equiv 1 \pmod{m}$). The inverse of an element modulo m exists if and only if the element is relatively prime to m.

Theorem 1.1.2 - Euler's theorem
Let $\phi(n)$ be the number of positive integers relatively prime to n. Let a be relatively prime to n. Then $a^{\phi(n)} \equiv 1 \pmod{n}$.

Theorem 1.1.3 - Fermat's little theorem
Let p be a prime number. Let a be relatively prime to p. Then $a^{p-1} \equiv 1 \pmod{p}$. It's a direct consequence of taking a prime n in Euler's theorem, since $\phi(n) = n - 1$ if and only if n is a prime number.

Theorem 1.1.4 - Wilson's theorem
The number n is prime if and only if $(n-1)! \equiv -1 \pmod{n}$.

Theorem 1.1.5 - Chinese remainder theorem
Let $m_1, \ldots, m_n$ be positive, different from 1 and pairwise relatively prime. Then for any $a_1, \ldots, a_n$ the system of linear congruences

$$x \equiv a_1 \pmod{m_1}, \quad \ldots, \quad x \equiv a_n \pmod{m_n}$$

has solutions, and any two such solutions are congruent modulo $m = m_1 \cdots m_n$.

1.2 Advanced theory

1.2.1 The order of an element modulo n

Given the numbers a, n such that $n > 1$ and $\gcd(a, n) = 1$, **the order of a modulo n** is the smallest number d for which $a^d \equiv 1 \pmod{n}$ (such a number exists, since from the Euler's theorem $a^{\phi(n)} \equiv 1 \pmod{n}$, but it doesn't have to be the smallest number with that property). If $\gcd(a, n) > 1$, the order of a modulo n doesn't exist. We denote the order of a modulo n as $\mathrm{ord}_n(a)$. The most important and powerful property of $\mathrm{ord}_n(a)$ is the fact that if $a^m \equiv 1 \pmod{n}$ for some m, then $\mathrm{ord}_n(a)$ divides m.

1.2.2 Primitive roots

We call g **a primitive root modulo** n if $\mathrm{ord}_n(g) = \phi(n)$. The primitive roots modulo n exists if and only if $n \in \{2, 4, p^\alpha, 2p^\alpha\}$, where $p \geq 3$ is any prime and α is any positive integer. If g is a primitive root modulo p, where p is an odd prime, then the sets $\{g, g^2, \ldots, g^{p-1}\}$ and $\{1, 2, \ldots, p-1\}$ are equal. Additionally, g is a primitive root modulo odd prime p if and only if $g^{\frac{p-1}{2}} \equiv -1 \pmod{p}$ and $g^k \not\equiv 1 \pmod{p}$ for all $1 \leq k \leq \frac{p-1}{2}$. Also, given an odd prime p and its primitive root g, g^k is a quadratic residue if and only if k is even.

1.2.3 Quadratic residues

Given relatively prime numbers a and $n > 0$, we call a **a quadratic residue modulo** n if the congruence $x^2 \equiv a \pmod{n}$ has a solution. Otherwise we call a **a quadratic nonresidue modulo** n. For p being an odd prime, there are $\frac{p-1}{2}$ quadratic residues in the set $\{1, 2, \ldots, p-1\}$. Let p be a prime and let a be a positive integer relatively prime to p. The **Legendre symbol** of a with respect to p is defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue (mod } p\text{)}, \\ -1 & \text{otherwise.} \end{cases}$$

The most useful properties of the Legendre symbol with respect to an odd prime p are as follows:

- If $a \equiv b \pmod{p}$, then $\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$.

- For any numbers $a_1, a_2, \ldots, a_k$ relatively prime to p, $\left(\frac{a_1 a_2 \cdots a_k}{p}\right) = \left(\frac{a_1}{p}\right)\left(\frac{a_2}{p}\right)\cdots\left(\frac{a_k}{p}\right)$.
- (**Euler's criterion**) $\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \pmod{p}$.
- (**The law of quadratic reciprocity**) Let p and q be two odd distinct primes. Then $\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}$.
- (**The first supplement to the law of quadratic reciprocity**) $\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}$.
- (**The second supplement to the law of quadratic reciprocity**) $\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$.

1.3 Other facts

Theorem 1.3.1 - Bézout's identity
For any numbers $a, b \neq 0$ denote $d = \gcd(a, b)$. Then there exist such numbers x, y that $ax + by = d$. The direct consequence of this is the fact that for every relatively prime numbers a, b there exist such numbers x, y that $ax+by = 1$.

Theorem 1.3.2 Let p be a prime number. Then $\binom{p}{i}$ is divisible by p for $1 \leq i \leq p-1$.

Theorem 1.3.3 Let $P(x)$ be a polynomial with integer coefficients and let n be positive integer. Let x, y be numbers such that $x \equiv y \pmod{n}$. Then $P(x) \equiv P(y) \pmod{n}$.

Theorem 1.3.4 Let p be a prime number and $P(x)$ be a polynomial of degree n with integer coefficients. Then the congruence

$$P(x) \equiv 0 \pmod{p}$$

has at most n incongruent solutions.

Theorem 1.3.5 - Thue's lemma
Let n be positive integer and a be a number relatively prime to n. Then there exist numbers x, y such that $|x|, |y| < \sqrt{n}$ and $x \equiv ay \pmod{n}$.

Theorem 1.3.6 - Dirichlet's theorem
Given relatively prime a, b, the sequence $an + b$ for $n = 0, 1, 2, \ldots$ contains an infinite number of primes.

Theorem 1.3.7 Let p be an odd prime. Then the congruence

$$ax^2 + bx + c \equiv 0 \pmod{p}$$

where $p \nmid a$ has a solution if and only if $\Delta \equiv 0 \pmod{p}$ or $\left(\frac{\Delta}{p}\right) = 1$ where $\Delta = b^2 - 4ac$. The solutions are given by this formula:

$$x \equiv \widetilde{2a}(-b \pm \sqrt{\Delta}) \pmod{p}$$

where $\sqrt{\Delta}$ is any number for which $(\sqrt{\Delta})^2 \equiv \Delta \pmod{p}$ and $\widetilde{2a}$ is the inverse (modulo p) of $2a$.

Theorem 1.3.8 For any prime $p = 4k + 1$ there exist integers x, y such that $p = x^2 + y^2$.

Theorem 1.3.9 Positive integer n is a sum of two squares of integers if and only if the power of every prime divisor $p \equiv 3 \pmod{4}$ in the factorisation of n is even.

2 Problems I

Problem 2.1 Let $P(x)$ be a polynomial with integer coefficients such that $n \mid P(2^n)$ for every positive integer n. Prove that $P(x) \equiv 0$.

Source: ELMO 2016

Problem 2.2 Let a_0 be a positive integer and $a_n = 5a_{n-1} + 4$ for all $n \geq 1$. Can a_0 be chosen so that a_{54} is a multiple of 2013?

Source: Baltic Way 2013

Problem 2.3 Prove that the sequence $\{2^n - 3 \mid n = 2, 3, \ldots\}$ contains an infinite subsequence whose members are all relatively prime.

Problem 2.4 Determine all positive integers relatively prime to all the terms of the infinite sequence

$$a_n = 2^n + 3^n + 6^n - 1, \ n \geqslant 1.$$

Source: IMO Shortlist 2005

Problem 2.5 Suppose that $\gcd(a, b) = 1$ and p is a prime. Prove that any prime factor q of $\frac{a^p - b^p}{a-b}$ is either equal to p or of the form $1 + kp$.

Problem 2.6 Find all integer solutions of the following equation:

$$\frac{x^7 - 1}{x - 1} = y^5 - 1.$$

Problem 2.7 Show that if m, n are positive integers, then $4mn - m - n$ cannot be a square of an integer.

Problem 2.8 Given a positive prime number p. Prove that there exist a positive integer α such that $p \mid \alpha(\alpha - 1) + 3$, if and only if there exist a positive integer β such that $p \mid \beta(\beta - 1) + 25$.

Source: Spain 2016

Problem 2.9 Let n be a positive integer with the following property: $2^n - 1$ divides a number of the form $m^2 + 81$, where m is a positive integer. Find all possible n.

Source: Hong Kong TST 2017

Problem 2.10 Let n be a positive integer. Prove that the number $2^n + 1$ has no prime divisor of the form $8k - 1$, where k is a positive integer.

Source: Vietnam TST 2003

Problem 2.11 Prove that if $n > 1$ then any prime factor of $2^{2^n} + 1$ is congruent to 1 (mod 2^{n+2}).

Problem 2.12 Let p be a prime number, and let n be a positive integer. Find the number of quadruples (a_1, a_2, a_3, a_4) with $a_i \in \{0, 1, \ldots, p^n - 1\}$ for $i = 1, 2, 3, 4$, such that

$$p^n \mid (a_1 a_2 + a_3 a_4 + 1).$$

Source: Baltic Way 2014

Problem 2.13 Let a, b be positive integers such that $b^n + n$ is a multiple of $a^n + n$ for all positive integers n. Prove that $a = b$.

Source: IMO Shortlist 2005

Problem 2.14 The sequence $\{a_n\}_{n \geq 0}$ is defined by $a_0 = 2, a_1 = 4$ and

$$a_{n+1} = \frac{a_n a_{n-1}}{2} + a_n + a_{n-1}$$

for all positive integers n. Determine all prime numbers p for which there exists a positive integer m such that p divides the number $a_m - 1$.

Source: MEMO 2016

Problem 2.15 Find all positive integers n such that for any integer k there exists an integer a for which $a^3 + a - k$ is divisible by n.

Source: APMO 2014

3 Problems II

Problem 3.1 Find all natural numbers m, n such that mn divides $(2^{2^n}+1)(2^{2^m}+1)$.

Source: Bulgaria 2016

Problem 3.2 Let p be an odd prime. Show that

$$1^i + 2^i + \ldots + (p-1)^i \equiv 0 \pmod{p} \quad \text{for} \quad 1 < i < p-1.$$

Problem 3.3 Prove that if $p = 4k+1$ is prime, then $p \mid k^k - 1$.

Problem 3.4 Let $p > 3$ be a prime such that $p \equiv 3 \pmod 4$. Given a positive integer a_0 define the sequence $a_0, a_1, \ldots$ of integers by $a_n = a_{n-1}^{2^n}$ for all $n = 1, 2, \ldots$. Prove that it is possible to choose a_0 such that the subsequence $a_N, a_{N+1}, a_{N+2}, \ldots$ is not constant modulo p for any positive integer N.

Source: Baltic Way 2016

Problem 3.5 Given are integers a, b such that $a \neq 0$ and $6a \mid 3 + a + b^2$. Prove that $a < 0$.

Source: Poland 2013

Problem 3.6 Let p, q be prime numbers (q is odd). Prove that there exists an integer x such that: $q|(x+1)^p - x^p$ if and only if $q \equiv 1 \pmod p$.

Source: Iran 2016

Problem 3.7 Prove that if a is a quadratic residue for every prime, then a is a square of an integer.

Problem 3.8 Prove that $2^{3^n}+1$ has at least n distinct prime divisors in the form $8k+3$.

Problem 3.9 Prove that $2^{2^n}+1$ has a prime divisor greater than $2^{n+2}(n+1)$.

Problem 3.10 Let $a > 1$ be a positive integer. Prove that there exist integer $n \geq 0$ such that $2^{2^n} + a$ is not prime.

Problem 3.11 Find all positive integers n such that there exists a unique integer a such that $0 \leqslant a < n!$ with the following property:

$$n! \mid a^n + 1.$$

Source: IMO Shortlist 2005

Problem 3.12 Let k be a fixed integer greater than 1, and let $m = 4k^2 - 5$. Show that there exist positive integers a and b such that the sequence (x_n) defined by

$$x_0 = a, \quad x_1 = b, \quad x_{n+2} = x_{n+1} + x_n \quad \text{for} \quad n = 0, 1, 2, \ldots,$$

has all of its terms relatively prime to m.

Source: IMO Shortlist 2004

Problem 3.13 Determine all integers $n > 1$ such that $\frac{2^n+1}{n^2}$ is an integer.

Source: IMO 1990

Problem 3.14 Let x and y be positive integers. If $x^{2^n} - 1$ is divisible by $2^n y + 1$ for every positive integer n, prove that $x = 1$.

Source: IMO Shortlist 2012

Problem 3.15 Prove that there does not exist positive integers a, b and k such that $4abk - a - b$ is a perfect square.

Source: Olympic Revenge 2017

Robert Crumplin & Paweł Piwek - Nilpotent Groups

Acknowledgement: the following class is based on the notes of the course "Topics in Groups" taught by Martyn Quick at University of St. Andrews in 2009.

0 Notation

- By $H \leqslant G$ we mean that H is a subgroup of G. When we write $H < G$ we mean that H is a subgroup of G distinct from G itself.
- By $H \trianglelefteq G$ we mean that H is a normal subgroup of G.
- By E we denote the trivial subgroup of a given group.
- If $H \trianglelefteq G$, by $v_H : G \twoheadrightarrow {}^{G}\!/_{H}$ we denote the natural quotient homomorphism induced by H.
- By $\mathcal{Z}(G)$ we denote the center of G, i.e. $\{h \in G | \forall g \in G : gh = hg\}$.
- If $H \leqslant G$, by $\mathcal{N}_G(H)$ we denote the normaliser of H in G, i.e. $\{g \in G | gHg^{-1} = H\}$.
- Let $A, B \subseteq G$, by $[A, B]$ we denote the commutator of A, B, which is the subgroup generated by elements of form $aba^{-1}b^{-1}$ where $a \in A, b \in B$.
- If p is a prime, by $\mathrm{Syl}_p(G)$ we mean the set of all Sylow p-subgroups of G.

1 Definitions and simple properties

Normal series carry important information about a given group. We may compare looking at subgroups to zooming our view of the group on the "surroundings" of the identity, whereas looking at quotients of the group to "changing our focus", while still looking at the whole group.

Central series carry information about how much "abelian structure" a given group has.

Definition 1.1. Let G be a group and $G = G_0 \geqslant G_1 \geqslant \ldots$ its normal series (i.e. for every i: $G_i \trianglelefteq G$). We call it a *central series* of a group G if:

$$\forall i : {}^{G_i}\!/_{G_{i+1}} < \mathcal{Z}\Big({}^{G}\!/_{G_{i+1}}\Big).$$

Definition 1.2. A group G is called *nilpotent* if it has a central series, which terminates at $G_n = E$ for some n.

Example 1.1. Here are a few examples of nilpotent groups:

- Every abelian group is nilpotent.
- Q_8, the quaternion group is nilpotent:

$$Q_8 > \{1, -1, i, -i\} > \{1, -1\} > \{1\} = E.$$

- D_8, the group of symmetries of a square is nilpotent. Let $D_8 = \langle r, h | r^4 = h^2 = e, hrh = r^{-1} \rangle$.

$$D_8 > \{e, r, r^2, r^3\} > \{e, r^2\} > \{e\} = E.$$

Out of that it is natural to define the "upper limit for central series" as below, with the origin of the name explained in the proposition following the definition.

Definition 1.3. Let G be a group. Define $E = Z_0 \leqslant Z_1 \leqslant \ldots$ as:

$$Z_{i+1} = v_{Z_i}^{-1}\big(\mathcal{Z}\Big({}^{G}\!/_{Z_i}\Big)\big),$$

where $v_{Z_i} : G \twoheadrightarrow {}^{G}\!/_{Z_i}$ is the natural homomorphism induced by the normal subgroup Z_i. In other words, Z_{i+1} is the union of cosets with respect to Z_i that commute with all other cosets.

This series is called the *upper central series* of a group G.

Proposition 1.1. *If $G = G_0 \geqslant G_1 \geqslant \ldots \geqslant G_m = E$ is a central series of a nilpotent group G and $E = Z_0 \leqslant Z_1 \leqslant \ldots$ is the upper central series of G, then for every i:*

$$G_{m-i} \leqslant Z_i.$$

It turns out that one can get a "lower limit for central series" as well.

Definition 1.4. Let G be a group. Define a subgroup series $G = K_0 \geqslant K_1 \geqslant \dots$ by:

$$K_{i+1} = [K_i, G].$$

This series is called the *lower central series* of the group G.

Proposition 1.2. *Let $G = G_0 \geqslant G_1 \geqslant \dots \geqslant G_n = E$ be a central series of a group G and $G = K_0 \geqslant K_1 \geqslant \dots$ its lower central series. Then for every i:*

$$K_i \leqslant G_i.$$

Proof. We need to show that if $K_i \leqslant G_i$, then $K_{i+1} = [K_i, G] \leqslant G_{i+1}$. It suffices to show that $[G_i, G] \leqslant G_{i+1}$, which is equivalent to the following:

$$\forall h \in G_i, g \in G: \quad v_{G_{i+1}}([h,g]) = e \iff v_{G_{i+1}}(h) v_{G_{i+1}}(g) v_{G_{i+1}}(h)^{-1} v_{G_{i+1}}(g)^{-1} = e.$$

By definition of central series, $v_{G_{i+1}}(G_i) \leqslant \mathcal{Z}\left({}^{G}/_{G_{i+1}}\right)$, so $v_{G_{i+1}}(h)$ is commuting with any $v_{G_{i+1}}(g)$. □

Corollary. *Let G be a group. Then the following conditions are equivalent:*

- *G is nilpotent.*
- *The lower central series terminates at $K_n = E$ for some n.*
- *The upper central series terminates at $Z_n = G$ for some n.*

Additionally, the lower and upper central series are of equal lenght.

Definition 1.5. For a nilpotent group G, we call the length of its upper (or equivalently - lower) central series the *nilpotency class* of G.

However, it is not true that the length of *any* central series of G is equal to the nilpotency class of G. In fact, in one of the problems you are asked to come up with a counterexample.

There are another two properties that are rather important:

Proposition 1.3. *Let G be nilpotent and $H \leqslant G$. Then H is nilpotent too.*

Proposition 1.4. *Let G be nilpotent and $\varphi : G \twoheadrightarrow H$ be a surjective homomorphism. Then H is nilpotent too.*

1.1 Problems

Problem 1.1. Give three examples of non-nilpotent groups. Can you give one that isn't a finite group?

Problem 1.2. Find a group G and its normal subgroup N such that both N and ${}^{G}/_{N}$ are nilpotent, but G itself isn't nilpotent.

Problem 1.3. Find a group G together with its central series, such that the series is of length different than the nilpotency class of the group.

Problem 1.4. Let G and H be groups. Show that $\mathcal{Z}(G \times H) = \mathcal{Z}(G) \times \mathcal{Z}(H)$. Then, show that the corresponding terms in the upper central series obey the following: $Z_i(G \times H) = Z_i(G) \times Z_i(H)$. Conclude that a direct product of nilpotent groups is nilpotent.

Problem 1.5. Prove that all finite p-groups are nilpotent.

2 Classification of finite nilpotent groups

We are first going to need two propositions:

Proposition 2.1. *Let G be a finite group, p be a prime number and $P \in \mathrm{Syl}_p(G)$. Then:*

$$\mathcal{N}_G(\mathcal{N}_G(P)) = \mathcal{N}_G(P).$$

Proof. Let's consider P as a Sylow p-subgroup of $\mathcal{N}_G(\mathcal{N}_G(P))$. By the second Sylow Theorem:

$$\forall Q \in \mathrm{Syl}_p\left(\mathcal{N}_G(\mathcal{N}_G(P))\right) : \exists x \in \mathcal{N}_G(\mathcal{N}_G(P)) : Q = xPx^{-1}.$$

On the other hand, if $x \in \mathcal{N}_G(\mathcal{N}_G(P))$, then $x\,\mathcal{N}_G(P)\,x^{-1} = \mathcal{N}_G(P)$, so we must have $Q \in \mathcal{N}_G(P)$. However, this means that all Sylow p-subgroups of $\mathcal{N}_G(\mathcal{N}_G(P))$ are actually Sylow p-subgroups of $\mathcal{N}_G(P)$, so again by second Sylow Theorem, they must be conjugate by some element of $\mathcal{N}_G(P)$. That means that there is only one Sylow p-subgroup of $\mathcal{N}_G(\mathcal{N}_G(P))$ as all elements of $\mathcal{N}_G(P)$ stabilize P when it comes to conjugation.

Now, let's consider the conjugation action of $\mathcal{N}_G(\mathcal{N}_G(P))$ on $\mathrm{Syl}_p(\mathcal{N}_G(P))$. The stabilizer in this case is the normalizer subgroup. This, by orbit-stabilizer theorem gives us:

$$1 = \left|\mathrm{Syl}_p\left(\mathcal{N}_G(P)\right)\right| = \left[\mathcal{N}_G(\mathcal{N}_G(P)) : \mathcal{N}_G(P)\right],$$

so we indeed have $\mathcal{N}_G(\mathcal{N}_G(P)) = \mathcal{N}_G(P)$. □

Proposition 2.2. *Let G be a finite nilpotent group. Then for any $H < G$:*

$$H < \mathcal{N}_G(H).$$

Proof. We'll prove it by induction on the nilpotency class of G. The base case is obvious, as groups of nilpotency class 1 are just abelian groups. Now, let's assume that the theorem is true for any groups G of nilpotency class at most n.

Obviously, for any $H \leqslant G : \mathcal{Z}(G) \leqslant \mathcal{N}_G(H)$. Now we have two cases:

- $\mathcal{Z}(G) \nleqslant H$ - than we clearly have some $z \in \mathcal{Z}(G)$ such that $z \notin H$, but $z \in \mathcal{N}_G(H)$.
- $\mathcal{Z}(G) \leqslant H$. In this case we look at ${}^{H}\!/_{\mathcal{Z}(G)} < {}^{G}\!/_{\mathcal{Z}(G)}$. The nilpotency class of ${}^{G}\!/_{\mathcal{Z}(G)}$ is one less than nilpotency class of G, so we can use our induction hypothesis. That means that:

  $$\mathcal{N}_{G/\mathcal{Z}(G)}\left({}^{H}\!/_{\mathcal{Z}(G)}\right) > {}^{H}\!/_{\mathcal{Z}(G)},$$

  so clearly $\mathcal{N}_G(H) > H$. □

Theorem 2.3. *A finite group is nilpotent if and only if it is a direct product of (some number of) p-groups.*

2.1 Problems

Problem 2.1. (Frattini argument) Let G be a finite group, $K \trianglelefteq G$ and $P \in \mathrm{Syl}_p(K)$. Then:

$$G = \mathcal{N}_G(P)K.$$

Use this lemma to get an alternative proof of Proposition 2.1.

Problem 2.2. Let G be a finite group. Prove that the following conditions are equivalent:

- G is nilpotent.
- Every Sylow p-subgroup is normal in G.
- Every maximal proper subgroup of G is normal in G.
- G is product of finite p-groups.
- For all $H < G :\ H < \mathcal{N}_G(H)$.

Robert Crumplin & Paweł Piwek - Simplicity

0 Notation

- By $H \leqslant G$ we mean that H is a subgroup of G. When we write $H < G$ we mean that H is a subgroup of G distinct from G itself.
- By $H \trianglelefteq G$ we mean that H is a normal subgroup of G.
- By E we denote the trivial subgroup of a given group.
- By C_n we denote the cyclic group of order n.
- By A_n we denote the alternating group - the group of all even permutations on n elements.
- If G acts on a set X and $x \in X$, by $\mathrm{Orb}_G(x)$ we denote the orbit of the element x, i.e. $\{y \in X | \exists g \in G : g(x) = y\}$.
- If $H \leqslant G$, by $\mathcal{N}_G(H)$ we denote the normaliser of H in G, i.e. $\{g \in G | gHg^{-1} = H\}$.

1 Definitions and examples

Definition 1.1. A group G is called *simple* if it has no non-trival normal subgroups.

Proposition 1.1. *If G is simple and $\varphi : G \to H$ is a homomorphism, then it must be either an injection, or the trivial homomorphism.*

Theorem 1.2. *The only nontrivial abelian simple groups are C_p where p is a prime.*

Now, we are going to see another family of simple groups - A_n. First, we'll prove a few facts about them.

Proposition 1.3. *A_n is generated by 3-cycles.*

Proof. As any even permutation is a product of an even number of transposition, it suffices to show that the product of any two transpositions is expressable as a product of 3-cycles. Indeed:

$$(ab)(ab) = (), (ab)(bc) = (abc), (ab)(cd) = (acb)(acd),$$

which finishes the proof. □

Proposition 1.4. *Let $n \geqslant 5$. If $H \trianglelefteq A_n$ is a normal subgroup which contains a 3-cycle then $H = A_n$.*

Proof. Without loss of generality, suppose $(123) \in H$. In S_n we may conjugate to any other 3-cycle:

$$(\sigma(1)\ \sigma(2)\ \sigma(3)) = \sigma(123)\sigma^{-1}.$$

The only question is if the permutation σ we conjugate by is itself even, i.e. we conjugate inside A_n. If it is, we are done. If it isn't, let's consider $\tau = \sigma(45)$. As (45) and (123) are disjoint, they commute, so:

$$\tau(123)\tau^{-1} = \sigma(45)(123)(45)\sigma^{-1} = \sigma(123)\sigma^{-1} = (\sigma(1)\ \sigma(2)\ \sigma(3)),$$

but τ and σ differ by a transposition, so one of them has to be even. □

Note that we had to have $n \geqslant 5$ to use (45).

Proposition 1.5. *Every normal subgroup $H \trianglelefteq A_n$ contains a 3-cycle.*

Proof. We are going to consider different cases for the longest cycle in the disjoint cycle decomposition types for given $\sigma \in H$.

- Suppose we have $\sigma = (123...r)\tau \in H$ for $r \geqslant 4$ and τ not containing 1,2 or 3. Then, by normality, the commutator of σ and (123) is in H. Upon expanding we see this is equal to $(23r)$.
- Suppose $\sigma = (123)(456)\tau \in H$ where τ doesn't have cycles of length more than 3. Calculating the commutator of σ with (123) gives $(123456) \in H$ which reduces to the previous case.
- Suppose $\sigma = (123)\tau$ for τ a product of 2-cycles. Squaring gives us our desired 3-cycle.
- Now suppose $(12)(34)\tau \in H$ where τ is a product of 2-cycles. Once again commutating with (123) gives $(14)(23) = u$. We calculate $(152)u(125) = (13)(45) = v \in H$. We then see $uv = (12345) \in H$ which is again covered by the first case.

□

Putting all these results together gives the following:

Theorem 1.6. *For $n \geqslant 5$ the alternating group A_n is simple.*

1.1 Problems

Problem 1.1. Find an example of a group G, which is simple and infinite.

2 Sylow's Theorems

> For a group theorist, Sylow's Theorem is such a basic tool, and so fundamental, that it is used almost without thinking, like breathing.
>
> *Geoff Robinson*

Definition 2.1. Let G be a finite group of size $n = p^\alpha m$ where p is prime and it doesn't divide m. Then if $H \leqslant G$ has size p^α it is called a *Sylow p-subgroup* of G. By $\mathrm{Syl}_p(G)$ we denote the set of all Sylow p-subgroups of G.

Theorem 2.1. ***(First Sylow's Theorem)*** *Let G be a finite group. Then, it has a Sylow p-subgroup for every prime p.*

Proof. Consider G acting on the family $\mathcal{F}$ of subsets of the set of elements of G of size p^α by left multiplication of the elements. Let $A = \{g_1, g_2, \ldots, g_{p^\alpha}\}$ be one of these subsets and $h \in G$ be a given element of G. Then, when we act on this subset by hg_1^{-1}, we get a subset $hg_1^{-1}A = \{h, hg_1^{-1}g_2, \ldots, hg_1^{-1}g_{p^\alpha}\}$. Therefore every element of the group G has to belong to some element of a given orbit (e.g. the orbit of A), so the size of any orbit is at least $\frac{|G|}{p^\alpha} = m$. If we had an orbit of size exactly m, then the size of the stabilizer of any of its elements would be p^α and we would indeed have a Sylow p-subgroup of G.

Let's assume otherwise - that $\forall A \in \mathcal{F} : |\mathrm{Orb}_G(A)| > m$, but that means that $\forall A \in \mathcal{F} : p \,|\, |\mathrm{Orb}_G(A)|$. However, $p \nmid \binom{p^\alpha m}{p^\alpha} = |\mathcal{F}|$, which we leave as an easy exercise. Thus, we get a contradiction. □

Theorem 2.2. ***(Second Sylow's Theorem)*** *Let G be a finite group. Then, for a given prime p, all Sylow p-subgroups of G are conjugate.*

We will prove a slightly stronger result using the following lemma:

Lemma 2.3. ***(Fixed-point congruence for p-group)*** *Let G be a p-group acting on a set X. Then the number of fixed points is congruent to the size of X modulo p.*

Theorem 2.4. *Let G be a group, $P \in \mathrm{Syl}_p(G)$ and Q be its any p-subgroup, where $|P| = p^\alpha$. Then:*

$$\exists g \in G : g^{-1}Qg \leqslant P.$$

Proof. Let Q act on the set of left cosets of P in G by left multiplication. Then, gP is a fixed point if and only if:

$$\forall q \in Q : qgP = gP \iff \forall q \in Q : g^{-1}qg \in P \iff g^{-1}Qg \leqslant P.$$

From fixed-point congruence for p-groups we know that as the size of the set acted on is $\frac{|G|}{p^\alpha}$, which isn't divisible by p, there must be some fixed points, so such g indeed exists. □

Theorem 2.5. ***(Third Sylow's Theorem)*** *Let G be a finite group, $|G| = p^\alpha m$, where $p \nmid m$. Let $n_p = |\mathrm{Syl}_p(G)|$. Then $n_p | m$ and $n_p \equiv 1 \mod p$.*

Proof. Let G act on $\mathrm{Syl}_p(G)$ by conjugation - we know that the action is transitive by Second Sylow's Theorem. Then, by orbit-stablizer theorem, for any $P \in \mathrm{Syl}_p(G)$:

$$n_p = [G : \mathcal{N}_G(P)] \;\Big|\; [G : P] = m,$$

as $P \leqslant \mathcal{N}_G(P)$.

Now, let P act on $\mathrm{Syl}_p(G)$ by conjugation. Then there is definitely one singleton orbit - P. We want to prove that there are no other singleton orbits to get by fixed-point congruence that $n_p \equiv 1 \mod p$. Assume otherwise, let $Q \in \mathrm{Syl}_p(G)$ satisfy the following:

$$\forall p \in P : p^{-1}Qp = Q \iff P \leqslant \mathcal{N}_G(Q).$$

However, we may look at P and Q as Sylow p-subgroups of $\mathcal{N}_G(Q)$, so by Second Sylow's Theorem they are conjugate by an element of $\mathcal{N}_G(Q)$. However, $\forall n \in \mathcal{N}_G(Q) : n^{-1}Qn = Q$, so we must have $P = Q$. □

3 Counterexamples

This section will be concerned with showing that a subgroup of a given size cannot be simple, i.e. it has to have a non-trivial normal subgroup. The main two techniques used will be third Sylow's Theorem and the following lemma:

Lemma 3.1. *Let G be a group acting transitively on a set of n elements, where $n > 1$ and $n! < |G|$. Then G is not simple.*

Proof. The action gives us a homomorphism $\varphi : G \to S_n$. Assuming G was simple, the homomorphism must be either trivial, or injective. It obviously can't be trivial as $n > 1$ and it can't be injective as $|G| > n!$, so we get a contradiction. □

Now it is time to use the techniques.

Example 3.1. Prove that no group G of order 75 is simple.

Proof. Let's see that $|G| = 3 \cdot 5^2$ and let's look at the Sylow 5-subgroups of G. The number of them must be a divisor of 3 congruent to 1 modulo 5. That means that there is only one Sylow 5-subgroup of G, so it has to be normal. □

Example 3.2. Let G be a group of order $351 = 3^3 \cdot 13$. Then G has a normal p-subgroup for some p.

Proof. $n_{13}|27$ and $n_{13} \equiv 1 \mod 13$, so $n_{13} = 1 \vee n_{13} = 27$. If $n_{13} = 27$, then there are $27 \cdot (13 - 1) = 324$ elements of order 13, so all of the other $351 - 324 = 27$ elements must form the Sylow 3-group. □

Example 3.3. Prove that no subgroup of order 300 is simple.

Proof. $300 = 2^2 \cdot 3 \cdot 5^2$, let's look at the Sylow 5-subgroups of G.

$$n_5|12 \quad \wedge \quad n_5 \equiv 1 \mod 5 \Rightarrow n_5 = 1 \vee n_5 = 6,$$

so if we assume that G is simple, we get that $n_5 = 6$. G acts transitively by conjugation on the set of its Sylow 5-subgroups, so by the lemma we've proved, G cannot be simple.

3.1 Problems

Problem 3.1. Show that every group G of size 105 has unique subgroups of size 5 and 7.

Problem 3.2. (Rather big one) Prove that only simple groups of size less than 100 are C_p for $p < 100$ prime and A_5.

Piotr Ambroszczyk - Projective Geometry

★ - means this fragment was shown during classes.

Definition. Transformation of plane is called *affine* if it is continuous, one-to-one and the image of every line is a line.

Firstly we will introduce example of affine transformation.

Definition. For a given line ℓ and a real number k we define *shear mapping* under which point P is transformed into point P' such that

$$\overrightarrow{HP} = k \cdot \overrightarrow{HP'}$$

where H is the projection of P onto line ℓ.

Obviously this transformation is continuous and one-to-one so we only have to prove that it preserves collinearity ★.

Property 1. Affine transformation maps parallel lines into parallel lines.

Proof. Affine transformation is one-to-one so the images of the lines can not intersect. □

Property 2. Let $A,\ B,\ C$ and D be distinct points on a plane such that $\overrightarrow{AB} = \overrightarrow{CD}$. Let f be an arbitrary affine transformation, denote $A' := f(A),\ B' := f(B),\ C' := f(C)$ and $D' := f(D)$, then $\overrightarrow{A'B'} = \overrightarrow{C'D'}$.

Proof. It comes from **Property 1**, affine transformation preserves parallelograms. □

This property allows us to define image of a vector under affine transformation $f(\overrightarrow{XY}) := \overrightarrow{f(X)f(Y)}$.

Property 3. Let f be an affine transformation, then

- $f(\overrightarrow{0}) = \overrightarrow{0}$
- $f(\overrightarrow{x} + \overrightarrow{y}) = f(\overrightarrow{x}) + f(\overrightarrow{y})$
- $f(k \cdot \overrightarrow{x}) = k \cdot f(\overrightarrow{x})$ where $k \in \mathbb{R}$.

Proof. Observe that $f(\overrightarrow{0}) = f(\overrightarrow{AA}) = \overrightarrow{f(A)f(A)} = \overrightarrow{0}$.

Using **Property 2** we can assume that $\overrightarrow{x} := \overrightarrow{AB}$ and $\overrightarrow{y} := \overrightarrow{BC}$, then

$$f(\overrightarrow{x} + \overrightarrow{y}) = f(\overrightarrow{AB} + \overrightarrow{BC}) = f(\overrightarrow{AC}) = \overrightarrow{f(A)f(C)} = \\ = \overrightarrow{f(A)f(B)} + \overrightarrow{f(B)f(C)} = f(\overrightarrow{AB}) + f(\overrightarrow{BC}) = f(\overrightarrow{x}) + f(\overrightarrow{y})$$

so we are done with second part.

Since we have shown that the function f is additive, we have $f(q \cdot \overrightarrow{x}) = q \cdot f(\overrightarrow{x})$ for all $q \in \mathbb{R}$ and all vectors $\overrightarrow{x}$. Let k be an irrational numbers and let k_n be a sequence of rational number converges to k, then

$$f(k \cdot \overrightarrow{x}) = f\left(\lim_{n\to\infty}(k_n \cdot \overrightarrow{x})\right) = \lim_{n\to\infty} f(k_n \cdot \overrightarrow{x}) = \lim_{n\to\infty}(k_n \cdot f(\overrightarrow{x})) = k \cdot f(\overrightarrow{x})$$

by continuousness of f. □

This proportion shows that every affine transformation is composition of a linear transformation and a translation.

Property 4. Let $A,\ B$ and C be distinct and collinear points. Denote by $A',\ B'$ and C' their images under affine transformation f respectively, then

$$\frac{AB}{BC} = \frac{A'B'}{B'C'}.$$[1]

Proof. It follows directly from the last point of the **Property 3**. □

Lemma 1. Given two not degenerated $\triangle ABC$ and $\triangle A'B'C'$. There exists unique affine transformation mapping $\triangle ABC$ into $\triangle A'B'C'$.

[1] It can be proven that affine transformation preserves not only ratio of a segments division but also ratio of a figures areas.

Proof. Denote $\overrightarrow{\epsilon_1} := \overrightarrow{AB}$, $\overrightarrow{\epsilon_2} := \overrightarrow{AC}$, $\overrightarrow{\epsilon_1'} := \overrightarrow{A'B'}$ and $\overrightarrow{\epsilon_2'} := \overrightarrow{A'C'}$. Since $\{\epsilon_1, \epsilon_2\}$ and $\{\epsilon_1', \epsilon_2'\}$ are two pairs of linearly independent vectors, we can consider f - linear transformation mapping ϵ_1 into ϵ_1' and ϵ_2 into ϵ_2' composited with translation by the vector $\overrightarrow{AA'}$.

This transformation is well defined, one-to-one and preserves collinearity ★. Moreover it can be proven that linear transformation of $\mathbb{R}^2$ is continuous[2], so f is an affine transformation. By the definition of f we have $f(A) = A'$, $f(B) = f(A + \overrightarrow{\epsilon_1}) = f(A) + f(\overrightarrow{\epsilon_1}) = A' + \overrightarrow{\epsilon_1'} = B'$, analogously $f(C) = C'$ so we are done.

Uniqueness follows from fact that in vector space each vector can be written as a sum of basis vectors in exactly one way ★.

□

Example. The median lines of a triangle intersect at the one point.

Lemma 2. A point P lies inside of $\triangle ABC$. There exists an affine transformation f mapping $\triangle ABC$ into $\triangle A'B'C'$ and point P into P' such that P' is the orthocenter of $\triangle A'B'C'$.

Proof. Denote $D := AP \cap BC$. Let B' and C' be the arbitrary points on the plane. Choose point D' on $B'C'$ such that $\frac{BD}{DC} = \frac{B'D'}{D'C'}$. Denote by ℓ line perpendicular to $B'C'$ and passing through the point D'. Using continuousness of function and Ceva's theorem we can prove existence of the point A' lying on line ℓ such that affine transformation mapping $\triangle ABC$ into $\triangle A'B'C'$ satisfies the conditions of lemma ★. □

Lemma 3. For any ellipse - ζ there exists affine transformation mapping ζ into a circle.

Proof. It is enough to use shear mapping with respect to the bisector of major axis ★. □

Problem 1. The points A_B and A_C divide segment BC on three equal parts, moreover the point A_B lies closer to B than A_C. Analogously define the points B_C, B_A, C_A and C_B. Denote $A_1 := BB_A \cap CC_A$ i $A_2 := BB_C \cap CC_B$ and analogously B_1, B_2, C_1 and C_2. Show that lines A_1A_2, B_1B_2 and C_1C_2 intersect at the one point.

Problem 2. The points D, E i F lie one the sides BC, CA i AB respectively such that $2AF = FB$, $2BD = DC$ and $2CE = EA$. On the segments DF and EF choose points G and H respectively such that $2GD = GF$ and $2FH = HE$. Show that $AFGH$ is parallelogram.

Problem 3. Ellipse ξ is inscribed in $\triangle ABC$. Denote by D, E and F the tangency points of ξ with BC, AC and AB. Show that lines AD, BE and CF are collinear.

Problem 4. Prove that an affine transformation preserves middle of a cubic.

Problem 5. A point P lies inside $\triangle ABC$ with centroid G. Denote by D, E, and F a reflections of the point P with respect to the midpoints of the sides BC, AC and AB respectively. Let Q be reflection of the midpoint of the segment PG with respect to the point G. The line ℓ goes through Q and intersects lines BC, AC and AB at the points X, Y and Z respectively. Prove that lines XD, YE and ZF intersect at the one point.

Definition. *Projective plane* is an extension of euclidean plane where every line ℓ has its point at infinity - l_∞ through which pass all lines parallel to l. Set of such defined points constitutes *line at infinity.*

Observe that

- every two lines on a projective plane intersect at the exactly one point
- for any two points exists exactly one line passing through them

Definition. Let π_1 and π_2 be two distinct projective planes in the 3-dimensional space. Choose point T lying neither on π_1 nor on π_2. The *central projection* of π_1 to π_2 with center T is the map that assigns intersection point of line TP with plane π_2 to every point P in plane π_1.

Suppose that ℓ is an intersection line of a plane parallel to π_2 and going through T with plane π_1. Notice that for any point P on line ℓ the line TP doesn't intersect plane π_2 in euclidean fashion and intersects in projective one! Image of this line constitutes line at infinity of plane π_2. Now we see that central projection is an one-to-one transformation between the protective planes.

Definition. *Projective transformation* is a composition of central projections and an affine transformation.

Basic properties of projective transformation ★.

- is bijection

[2] $\mathbb{R}^n$ also

- preserves collinearity
- preserves concurrency
- transforms lines into lines
- maps conics into conics

Property 5. Let points $A,\ B,\ C$ and D lie on a one line. Denote by $A',\ B',\ C'$ and D' their images under projective transformation $\mathcal{P}$, then

$$(A, B; C, D) = (A', B'; C', D').$$

Proof. First of all central projection preserves cross ratio. Since an affine transformation preserves ratio, it preserves cross ratio as well. Projective transformation is their composition so we are done. □

Lemma 4. For any given line ℓ on a plane there exists projective transformation sending ℓ into the line at infinity.

Proof. It is enough to find central projection sending ℓ into the line at infinity ★. □

Example. Given is $\triangle ABC$ and point P such that P doesn't lie on the lines $BC,\ CA$ i AB. Denote $D := AP \cap BC,\ E := BP \cap AC,\ F := CP \cap AB,\ Q := AD \cap EF$ and $T := BC \cap EF$. A line passing through T intersects lines AC and AB at the points X and Y respectively. Show that lines $QX,\ FC$ and YD intersect at the one point.

Proof. Use lemma and send line XY to infinity. Now notice that lines QX and YD pass thorough the midpoint of segment FC ★. □

Quick classification of conic section

- parabola - conic tangent to the line at infinity
- hyperbola - conic intersecting the line at infinity in exactly two different points
- ellipse - conic disjoint with the line at infinity

Lemma 5. Let ζ be a non-degenerate conic section. There exists projective transformation mapping ζ into a circle.

Proof. Choose line ℓ disjoint with ζ and consider projective transformation sending ℓ into the line at infinity. Conic will be transformed into ellipse which can be mapped into circle using affine transformation. □

Definition. Points lie in a *general position* on a plane if no three of them are collinear.

Lemma 6. Points $\{A, B, C, D\}$ and $\{A', B', C', D'\}$ are two sets of points laying in a general position. There exists projective transformation $\mathcal{P}$ such that $\mathcal{P}(A) = A',\ \mathcal{P}(B) = B',\ \mathcal{P}(C) = C'$ and $\mathcal{P}(D) = D'$.

Proof. Denote by $E := AD \cap BC$ and $F := AB \cap CD$. Transforming line EF into the line at infinity, quadrilateral $ABCD$ become parallelogram. Compose it with affine transformation mapping this parallelogram into square. Since any projective transformation has its inverse the proof of existence of $\mathcal{P}$ is complete ★.

Now we have to prove uniqueness. First of all, notice that three points lying on a line ℓ and given cross ration determinate unique fourth point laying on ℓ. Moreover, any point T can be represented by pair of lines - $\{TE, TF\}$. Now we just look at the image of points $TE \cap AB$ and $TF \cap AD$ ★. □

Example. Given is $\triangle ABC$ and a point P not belonging to any of the lines $BC,\ CA$ and AB. Denote $A_1 := AP \cap BC,\ B_1 := BP \cap CA$ and $C_1 := CP \cap AB$. Choose a point Q not belonging to any of the lines $B_1C_1,\ C_1A_1$ and A_1B_1. Denote $A_2 := A_1Q \cap B_1C_1,\ B_2 := B_1Q \cap C_1A_1$ and $C_2 := C_1Q \cap A_1B_1$. Show that lines $AA_2,\ BB_2$ and CC_2 intersect at the one point.

Proof. Consider transformation preserving $\triangle ABC$ and sending P into its centroid. Afterwards, thesis follows from homothety and isotomic conjugation ★. □

Lemma 7. Let a point P lies inside circle ω. There exists projective transformation preserving ω and mapping P into the center of ω.

Proof. Use projective transformation to send polar of the point P with respect to ω into the line at infinity. Afterwards, make circle using affine transformation. □

Example. Let Γ be the circumcircle of $\triangle ABC$. Prove that exists projective transformation preserving Γ and mapping $\triangle ABC$ into equilateral triangle.

Proof. Using fact that projective transformation preserves tangency of lines and conics consider projective transformation mapping Lemoine point of $\triangle ABC$ into the center of Γ ★. □

Problem 6. Using projective transformation prove following theorems and lemmas

- Pappus theorem
- Desargue's theorem
- existence of Newton line in of a quadrilateral
- existence of Gergonne's point
- Steinbart lemma

Problem 7. The diagonals of quadrilateral $ABCD$ intersect at the point K. A line ℓ intersects lines AB, CD, BC and DA at the points X_1, X_2, X_3 and X_4 respectively. Denote $Y_1 := KX_1 \cap CD$, $Y_2 := KX_2 \cap AB$, $Y_3 := KX_3 \cap AD$ and $Y_4 := KX_4 \cap BC$. Show that points Y_1, Y_2, Y_3 and Y_4 are collinear.

Problem 8. Given is $\triangle ABC$ and points D, E i F lying on lines BC, CA and AB respectively such that, lines AD, BE and CF intersect at a point P. Internal bisector of $\sphericalangle PEA$ intersects internal bisector of $\sphericalangle PFA$ at the point X. Analogously define points Y and Z. Show that lines AX, BY and CZ intersects at the one point.

Problem 9. Circle ω is inscribed into quadrilateral $ABCD$. Denote by K, L, M and N tangency point of ω with sides AB, BC, CD and DA. Denote $X := AM \cap NK$ and $Y := BM \cap KL$. Show that lines DC, NL and XY are concurrent.

Problem 10. Circle ω is inscribed into $\triangle ABC$. Denote tangency point of ω and BC by D. Choose a point X laying on segment AD and inside of ω. Segments BX and CX intersect ω at the points P and Q respectively. Prove that lines BQ and CP intersect on line AD.

Problem 11. Circle ω into inscribed in quadrilateral $ABCD$. A point P lies on the diagonal AC. Segments PA, PB, PC and PD intersect ω at the points A', B', C' and D' respectively. Prove that quadrilateral $A'B'C'D'$ is harmonic.

Radomił Baran - Projective Geometry

Definition. *Projective range* is a set of points in projective geometry. A projective range may be a projective line or a conic. We denote a projective range of points X on a line p as $p(X)$.

Definition. *Projectivity* between two lines p and q is a bijective transformation $\phi : p \to q$ that preserves cross-ratio, i.e.

$$(A_1, A_2; A_3, A_4) = (\phi(A_1), \phi(A_2); \phi(A_3), \phi(A_4)),$$

where $A_i \in p$ and $\phi(A_i) \in q$.

Definition. *Perspectivity* between two lines p and q with center O is a transformation that maps point A on p to point B on q and points O, A and B are colinear.

We say that $p(X)$ and $q(Y)$ are *projective* if there exist a projectivity between these ranges, and we denote $p(X) \overline{\wedge} q(Y)$ or $p(X) \mapsto q(Y)$.
Similarly, $p(X)$ and $q(Y)$ are *perspective* if there exist a perspectivity between these ranges, and we denote $p(X) \overline{\overline{\wedge}} q(Y)$.

Property 1. *Let $p(X) \mapsto q(Y)$. Then, $p(X) \overline{\overline{\wedge}} q(Y)$ if and only if the intersection of p and q is fixed in this projectivity.*

Property 2. *Projectivity between two ranges is uniquely determined by three pairs of corresponding points.*

Definition. *Pencil of lines* is the set of lines that pass through a given point in a projective plane. We can also have another pencils (e.g. pencils of conics). We denote pencil of lines p that pass through a point A as $A(p)$.

Definition. *Projectivity* between two pencils $A(p)$ and $B(q)$ is a bijective transformation $\phi : p \to q$ that preserves cross-ratio, i.e.

$$(p_1, p_2; p_3, p_4) = (\phi(p_1), \phi(p_2); \phi(p_3), \phi(p_4)),$$

where $p_i \in A(p)$ and $\phi(p_i) \in A(q)$.

Definition. *Perspectivity* between two pencils $A(p)$ and $B(q)$ with axis l is a transformation that for any point $X \in l$ maps the line AX to the line BX.

We say that $A(p)$ and $B(q)$ are *projective* if there exist a projectivity between these pencils, and we denote $A(p) \overline{\wedge} B(q)$ or $A(p) \mapsto B(q)$.
Similarly, $A(p)$ and $B(q)$ are *perspective* if there exist a perspectivity between these pencils, and we denote $A(p) \overline{\overline{\wedge}} B(q)$.

Property 1. *Let $p(X) \mapsto q(Y)$. Then, $A(p) \overline{\overline{\wedge}} B(q)$ if and only if the line AB is fixed in this projectivity.*

Property 2. *Projectivity between two pencils is uniquely determined by three pairs of corresponding lines.*

The main idea of duality is that we interchange points with lines so that: "three points are colinear changes" to "three lines are concurrent". We have also that the point on a conic interchange with the tangent to this conic at this point.

Lemma 0.2. *Given two ranges of points A and B on lines k and l respectively. Suppose that $k(A) \mapsto l(B)$ and that $X_{\infty k} \mapsto X_{\infty l}$. If we take any angles α and β then set of all points X such that $\sphericalangle(XA, k) = \sphericalangle(XB, l)$ is a line.*

Proof. Let p_∞ be the line at infinity. Notice that all lines XA pass through the point at infinity $X_{\infty\alpha}$ and all lines XB pass through the point at infinity $X_{\infty\beta}$. Because $k(A) \mapsto l(B)$ we have $X_{\infty\alpha}(A) \mapsto X_{\infty\beta}(B)$. So it suffices to prove that this map is a perspectivity. But we only need to check if the line p_∞ is fixed, but it is obvious due to $X_{\infty k} \mapsto X_{\infty l}$ ★.

Example. Given a ΔABC and its orthocenter H. ω is the circumcircle of triangle BHC. We take a point P on ω. Lines BP and CP intersect lines AC and AB at points X and Y respectively. Show that when P moves on ω, circumcircles of quadraliterals $AXPY$ pass through one point different from A.

Proof. It's easy to see that when P moves on ω then ranges X and Y are projective. So $AC(X) \mapsto AB(Y)$. Now, taking P to be the reflection of A with respect to the midpoint of BC we have that $X \equiv X_{\infty AC}$ and $Y \equiv X_{\infty AB}$. So $X_{\infty AC} \mapsto X_{\infty AB}$. So now from lemma we have that perpendiculars to AC at X and to AB at Y intersect at point S and S moves along a line. So also the midpoint of AS moves along a line. But it is the circumcenter of $AXPY$ so now we have that all circumcircles of $AXPY$ pass through the reflection of A with respect to this line. So we are done.

Problem 0.3. Given a ΔABC and a point P on the line BC. The circle with diameter BP intersects the circumcircle of APC again at point Q. Let M be the intersection of PQ and AC, and H be the orthocenter of ΔABP Prove that when P varies on BC, MH passes through a fixed point.

Problem 0.4. Given a ΔABC and two isogonal points P and Q. Points D, E and F are intersections of lines AP, BP and CP with sides BC, AC and AB respectively. Let O be the circumcenter of ΔABC. Let X be the intersection of the line perpendicular to EF, passes through A and the line OD. Prove that $QX \perp BC$.

Problem 0.5. Given a ΔABC and a point D on the side AB. Let I be a point inside ΔABC on the angle bisector of ACB. The second intersections of lines AI and CI with the circumcircle of ACD are P and Q respectively. Similarly, the second intersection of lines BI and CI with the circumcircle of BCD are R and S respectively. Show that if $P \neq Q$ and $R \neq S$, then lines AB, PQ and RS pass through one point or are parallel.

Problem 0.6. Given a ΔABC and its circumcircle τ. The angle bisector of BAC intersects the side BC at D and τ at L. Point E is the reflection of D with respect to the midpoint of BC. Let ω be a circle passes through B and C. Let perpendicular lines through D and E intersect ω at X and Y respectively. Lines AX and BY intersect τ at P and Q respectively. Prove that PQ is a diameter of τ.

★ - means this fragment was shown during classes.

Theorem 1. (Steiner conic) *Given two pencils $A(p)$ and $B(q)$ of lines at two points A and B and a projective but not perspective mapping π of $A(p)$ onto $B(q)$. Then the intersection points of corresponding lines form a non-degenerate conic.*

Proof. Take three lines p_1, p_2 and p_3 that pass through A and three lines q_1, q_2 and q_3 that pass through B. Then we denote $X_1 := p_1 \cap q_1$, $X_2 := p_2 \cap q_2$ and $X_3 := p_3 \cap q_3$. Then we take the conic $\mathcal{C}$ passes through A, B, X_1, X_2, X_3. Then, if we have any point X on $\mathcal{C}$ we have that $(AX_1, AX_2; AX_3, AX) = (BX_1, BX_2; BX_3, BX_4)$. But $A(p) \mapsto B(q)$ so AX maps to BX and we are done.

Remark. If π is a perspectivity, then intersection points of corresponding lines lie on a line.

Theorem 2. (Dual of Steiner Conic) *Given two ranges $p(A)$ and $p(B)$ of points on two lines p and q and a projective but not perspective mapping π of $p(A)$ onto $p(B)$. Then the lines connecting corresponding points form a dual non-degenerate conic (The lines AB are tangent to this conic).*

Proof: ★

Remark. If π is a perspectivity, then the lines connecting corresponding points intersect at one point.

Problem 0.7. Let $\Delta T_A T_B T_C$ be extangents triangle of ΔABC. Let D be an arbitrary point on the line $T_B T_C$. The tangent (different from $T_B T_C$) from D to C-excircle intersects $T_C T_A$ at E. The tangent (different from $T_B T_C$) from D to B-excircle intersects $T_A T_B$ at F. Prove that the line EF is tangent to A-excircle.

Problem 0.8. Given an acute ΔABC and its Euler line k. Points M and N are reflections of B and C with respect to k. P is an arbitrary point on k. Point E is the intersection of lines PM and AC and F is the intersection of lines PN and AB. Let H be the orthocenter of ΔABC and S be its reflection with respect to the line EF. Prove that S lies on the circumcircle of ΔABC.

Problem 0.9. Given a ΔABC and two points P and Q. Now $\{D\} = PQ \cap BC$, $\{E\} = PQ \cap AC$ and $\{F\} = PQ \cap AB$. The parallel to AQ through D intersects AP at X. Similarly we define points Y and Z. Show that the points X, Y and Z are colinear.

Maciej Dziuba - Radical axis

Problem 1. Points A, B, C, D lie on the line l in this order. Circles ω_C , ω_B with diameters respectively AC, BD intersect at points X, Y. Point P lies on the line XY, but not on l. Denote by M, N the intersection of respectively CP with ω_C and BP with ω_B. Prove that the lines AM, DN, XY are corcurrent.

Problem 2. Circles ω_1, ω_2 intersect at points X, Y. Two common tangents to these circles intersect at the point P. Line l passing through P and intersect ω_1 at points A, C and ω_2 at points B, D and A, B, C, D lie on l in this order. Prove that tangent to ω_1 in C and tangent to ω_2 in B intersect on the line XY.

Problem 3. Let H be the orthocenter of an acute triangle ABC. The circle with center in the midpoint of the segment BC and passing through H intersects the side BC at A_1, A_2. Similarly, define the points B_1, B_2, C_1, C_2. Prove that the points $A_1, A_2, B_1, B_2, C_1, C_2$ are concyclic.

Problem 4. Let l_1 and l_2 be the parallel lines and points P, Q lie between them. For point A, which lies on l_1, let A_1 be the intersection of AP and l_2. Now, let A_2 be the intersection of A_1Q and l_1. For point $B \in l_1$ similarly define B_1 and B_2. Prove that the second point of intersection $\bigcirc(PBA_2)$ and $\bigcirc(PAB_2)$ lies on the line PQ.

Problem 5. Points D, E lie on the sides respectively AB, AC of $\triangle ABC$, such that $DE \parallel BC$. Point P lies inside $\triangle ADE$. Segments BP, CP intersect segment DE at points F, G. The circumcircles of the triangles DPG, FPE intersect at points P and Q. Prove that points A, P, Q are collinear.

Problem 6. Let ABC be a acute triangle with incenter I. Points E, F are midpoints of shorter arcs $\widehat{AC}, \widehat{AB}$ respectively (on $\bigcirc(ABC)$). Segment EF intersects the sides AB, AC at points P, Q respectively. Point D satises the conditions $PD \parallel BI, QD \parallel CI$. Let T denote the intersection of BF and CE. Prove that T, I, D are collinear.

Problem 7. Cyclic quadrilateral $ABCD$ has no parallel sides. Prove that the locus of points P, which satisfies condition $\sphericalangle DAP + \sphericalangle CBP = \sphericalangle CPD$ is a circle.

Problem 8. Circles o_1, o_2 intersect each other at two points. Circles ω_1, ω_2 are externally tangent to o_1 at A_1, A_2, internally tangent to o_2 at B_1, B_2 and intersect each other at points C, D. Prove that the lines A_1B_1, A_2B_2 and CD are concurrent.

Problem 9. Let the incircle and $A-$excircle be tangent to side BC at point D, E respectively. Circle ω_B is the reflection of circle inscribed in $\triangle ABE$ in the midpoint of segment AB. Circle ω_C is defined similarly. Prove that D lies on radical axis of ω_B and ω_C.

Problem 10. Let ω_1, ω_2 be the two circles. Line l_1 is tangent to ω_1 in A and to ω_2 in B, such that these circles lie on the same side of l_1. Line l_2 is tangent to ω_1 in C and to ω_2 in D, such that these circles lie on the different sides of l_2. Prove that intersection point of AC and BD lies on the line passing through the centers of ω_1 and ω_2.

Problem 11. Points A, B lie outside the circle ω. Line l passing through point A intersects ω at P, Q and BP, BQ intersect ω at S, R. Prove that all lines RS, for every line l, passing through a fixed point.

Problem 12. Point S lies inside the circle ω. Circles ω_1, ω_2 are externally tangent to each other at S and internally tangent to ω. External common tangents to ω_1 and ω_2 intersect at point P. Prove that all points P lie on one line.

Problem 13. Circle ω is tangent to two parallel lines l_1, l_2. Circle ω_1 is tangent to l_1 at point A and externally tangent to ω at B. Circle ω_2 is tangent to l_2 at point C, externally tangent to ω at D and to ω_1 at E. Lines AD and BC intersect in S. Prove that $SB = SD = SE$.

Problem 14. Consider four lines (No three concurrent and no two parallel). We have four ways to choose three of them, which bound the triangle. Prove that orthocenters of these four triangles are collinear. Line passing through these orthocenters is called Aubert's line.

Problem 15. Let P be the intersection of diagonals AC and BD in cyclic quadrilateral $ABCD$. Point X lies inside the quadrilateral and satisfies the condition $\sphericalangle XAB + \sphericalangle XCB = \sphericalangle XBC + \sphericalangle XDC = 90^\circ$. Prove that line PX passing through the center of $\bigcirc(ABCD)$. (If there exist the circle insribed in ABCD, then P is its center and we have the very useful lemma)

Problem 16. Let ABC be the triangle with $BAC = 90^\circ$. Point D is foot of altitude from A. Point X lies on the segment AD. Point K is on the segment BX and $CA = CK$. Point L is on the segment CX and $BA = BL$. Let P be the intersection of BL and CK. Prove that $PK = PL$.

Problem 17. Let ABC be a triangle with orthocenter H, circumcenter O and K, M, N are the midpoints of sides BC, CA, AB respectively. The line tangent to $\bigcirc(ABC)$ at A intersects MN at P. Let E, F be the feet of altitudes from B, C in $\triangle ABC$. Let T denote the intersection of the ray $\overrightarrow{KH}$ and $\bigcirc(ABC)$. Lines AT and EF intersect at point Q. Prove that $OH \perp PQ$.

Problem 18. Let O, H denote respectively the circumecenter and orthocenter of $\triangle ABC$. Lines BH, CH intersect $\bigcirc(ABC)$ at points E, F respectively. Segment EO intersects side AC at P and segment FO intersects side AB at Q. Tangents to $\bigcirc(ABC)$ in B, C intersect at point T. Prove that $TH \perp PQ$.

Problem 19. Point D lies on the side BC of $\triangle ABC$. Circle ω_1 is tangent to (ABC), segment AD and the segment BD at P. Circle ω_2 is tangent to (ABC), segment AD and the segment CD at Q. Let M be the midpoint of arc $\widehat{BC}$ in $\bigcirc(ABC)$ and N is the midpoint of the segment PQ. Let I denote the incenter of $\triangle ABC$ and K is the midpoint of the segment DI. Prove that M, N, K are collinear.

Problem 20. Let E, F be the tangency point of the circle ω, inscribed in $\triangle ABC$, and the sides respectively AC, AB. Point G is intersection of BE and CF. Let P, Q be the points, such that the quadrilaterals $BCEP$ and $CBFQ$ are parallelograms. Prove that $GP = GQ$.

Problem 21. Let ABC be a triangle and let I and O denote its incenter and circumcenter respectively. Let ω_A be the circle through B and C which is tangent to the incircle of the triangle ABC. The circles ω_B and ω_C are defined similarly. The circles ω_B and ω_C meet at a point A' distinct from A; the points B' and C' are defined similarly. Prove that the lines AA', BB' and CC' are concurrent at a point on the line OI.

Problem 22. Let I and H be the incenter and orthocenter of $\triangle ABC$. Let P be the point on BC, which satisfies $PI \perp AI$, and let M be the midpoint of AP. Prove that H lies on the radical axis of circle with diameter IM and the inscribed circle.

Gábor Damásdi - Finite Geometries

Problems

(a) How many points can we choose from a 4×6 grid, such that no four of the chosen points form the vertices of an axis parallel rectangle?

(b) How many points can we choose from a the vertices of regular 13-gon such that no distance occurs more than once between the chosen points?

(c) Ulti is a Hungarian card game where each game is played by three players. We would like to organize am ulti competition for seven players such that every player plays against every other player at least once. Organize the competition with the smallest possible number of games!

(d) We would like to organize an ulti competition for $3 < n < 9$ players. We would like achieve that every player plays against every other player exactly twice and no games are repeated. For which n-s is this possible?

(Hungarian National Math competition 2011)

(e) We have chosen k vertices of a regular n-gon such that every possible distance occurs exactly once between the chosen points. Express n from k.

(f) How many points can we choose from a $k \times \binom{k}{2}$ grid, such that no four of the chosen points form the vertices of an axis parallel rectangle?

(g) Find as many interesting properties of the following deck of cards as you can.

1	2
5	15
17	

2	3
6	16
18	

3	4
7	17
19	

4	5
8	18
20	

5	6
9	19
21	

6	7
10	20
1	

7	8
11	21
2	

8	9
12	1
3	

9	10
13	2
4	

10	11
14	3
5	

11	12
15	4
6	

12	13
16	5
7	

13	14
17	6
8	

14	15
18	7
9	

15	16
19	8
10	

16	17
20	9
11	

17	18
21	10
12	

18	19
1	11
13	

19	20
2	12
14	

20	21
3	13
15	

21	1
4	14
16	

This card set is connected to some of the previous problems. Try to find the connections!

A **projective plane** consists of a set of lines, a set of points, and a relation between points and lines called incidence, having the following properties:

(i) Given any two distinct points, there is exactly one line incident with both of them.

(ii) Given any two distinct lines, there is exactly one point incident with both of them.

(iii) Every line is incident with at least 3 points.

(iv) Every point is incident with at least 3 lines.

A projective plane is called **finite** if the set of the points is a finite set.

(h) In a finite projective plane each line is incident to the same amount of points.

(i) In a finite projective plane each point is incident to the same amount of lines.

If a plane has $q+1$ points on each of its lines, then it is called a projective plane of **order q**.

(j) Show that if a finite projective plane has $q+1$ points on one of its lines, then it has q^2+q+1 points in total, and it has q^2+q+1 lines.

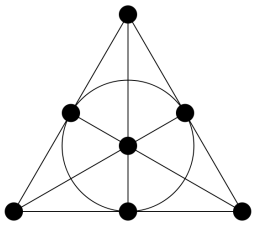
(k) We have chosen k points of a projective plane of order q such that we have chosen at least one point from each line. Show that $k \geq q+1$, and $k = q+1$ if and only if we have chosen the points of a single line.

(l) We have chosen k points of a projective plane of order q such that no three of them lies on a line. Show that $k \leq q+2$ if q is even and $k \leq q+1$ if q is odd.

(m) We have chosen $q+1$ points of a projective plane of order q such that no three of them lies on a line. Show that if q is even, then we can extend our set to $q+2$ points such that no three of them lies on a line.

(n) We have colored each point of a projective plane of order q with red or blue. Show that there is a line where the difference between the number of the red and blue points is at least $\sqrt{q}$.

(o) The Dobble game models the projective plane of order 7. Each object represents a point in the plane and each card represents a line. A point is incident to a line if and only if the corresponding object is drawn on the corresponding card. Two of the cards are missing from the deck. How can you reconstruct the missing cards quickly?



Projective plane of order 2. Also known as the Fano plane.

Andrzej Grzesik - From tic-tac-toe to Ramsey Theory

Some interesting facts

Theorem 1 (strategy stealing)**.** *In any symmetric game, where additional move is not a disadvantage, the second player cannot have a winning strategy.*

Tic-tac-toe 3-in-a-line game is a draw on 3×3 board, and a win on 3×4 and bigger boards. Tic-tac-toe 4-in-a-line game is a draw on boards up to 4×8 and 5×5. It's a win on 4×9, 5×6 and bigger boards. Tic-tac-toe 5-in-a-line game is a draw on boards up to 6×6, win on 15×15 and bigger boards. Boards between are undecided. Tic-tac-toe 8-in-a-line game is a draw on boards of any size. Games 6-in-a-line and 7-in-a-line are undecided.

Consider a set of words of length d over an alphabet with n letters. A *variable word* is a word over this alphabet with additional letter $\star$. A *combinatorial line* is a set of n words obtained by replacing all instances of $\star$ in a single variable word by each letter from the alphabet.

To illustrate, consider a set of words of length 2 over the alphabet $\{1, 2, 3\}$, i.e., {11, 12, 13, 21, 22, 23, 31, 32, 33}. Examples of combinatorial lines are $1\star = \{11, 12, 13\}$, $\star 3 = \{13, 23, 33\}$, $\star\star = \{11, 22, 33\}$.

Theorem 2 (Hales-Jewett)**.** *For any numbers k and c there exists a dimension d such that each hypercube k^d colored with c colors contains a monochromatic combinatorial line.*
Equivalently, each c-coloring of d-letter words over k-letter alphabet contains a monochromatic combinatorial line.

The smallest such number d is denoted by $HJ(k, c)$. Some known values and bounds:
$HJ(k, 1) = 1$, $HJ(2, c) = c$
$HJ(3, 2) = 4$ (Hindmann, Tressler 2014)
$HJ(4, 2) < 2^{2^{2^{2^{2^2}}}}$ (Shellah 1988) $\quad HJ(4, 2) < 10^{11}$ (Lavrov 2015)
$HJ(4, 2) > 11$ (Polymath 2009)
$HJ(k, 2) > \frac{2^{k/4}}{3k^4}$ (Beck, Pegden, Vijay 2009)
$HJ(3, 3) > 13$, $HJ(3, 4) > 17$, $HJ(3, 5) > 84$, $HJ(4, 3) > 97$, $HJ(5, 2) > 59$ (Polymath 2009)

Theorem 3 (Van der Waerden)**.** *For any k and c there exists n such that among numbers $1, 2, 3, \ldots, n$ colored on c colors, there exists a monochromatic arithmetic sequence of length k.*

The smallest such number n is denoted by $W(k, c)$. Some known values and bounds:
$W(3, 2) = 9$, $W(3, 3) = 27$, $W(3, 4) = 76$, $W(4, 2) = 35$, $W(5, 2) = 178$, $W(6, 2) = 1132$
$W(3, c) \leq e^{c^a}$, $W(4, c) \leq e^{e^{e^{c^a}}}$ (Shelah 1988)
$W(k, c) \leq e^{e^{c^{e^{e^{k+110}}}}}$ (Gowers 2001)

Theorem 4 (Szeméredi)**.** *For any number k and density d ($0 < d < 1$) there exists n such that any subset of $1, 2, 3, \ldots, n$ of size at least dn contains an arithmetic sequence of length k.*

Theorem 5 (Ramsey)**.** *For any numbers k and l, there exists n such that every graph on n vertices contains a complete graph of size k or an independent set of size l.*

The smallest such number n is denoted by $R(k, l)$. Some known values and bounds:
$R(3, 3) = 6$, $R(4, 4) = 18$, $R(4, 5) = 25$, $R(3, 9) = 36$, $43 \leq R(5, 5) \leq 49$, $102 \leq R(6, 6) \leq 165$.

Theorem 6 (Ramsey)**.** *For any numbers c and k, there exists n such that every c-coloring of edges of a complete graph on n vertices contains a monochromatic complete graph of size k.*

Problems

1. Find a winning strategy in 3×4 tic-tac-toe game.

2. Find a drawing strategy in 8-in-a-line tic-tac-toe game

3. Find a winning strategy in 3-in-a-line tic-tac-toe game on a dynamic 3×3 board.

4. Find a winning strategy in a 2×2 game (each player wants to create a 2×2 rectangle).

5. Find a winning strategy in ultimate tic-tac-toe game, where it is allowed to send opponent to already decided square.

6. Prove that $HJ(2, c) \leq c$.

7. Prove that $HJ(2, c) \geq c$.

8. Prove that $HJ(n, c) \geq n$.

9. Prove that for each integer c there exists an integer N such that if a board $N \times N$ is c-colored, then there are 4 monochromatic pieces forming a square.

10. Prove that for any given shape S and each integer c there exists an integer N such that if a board $N \times N$ is c-colored, then there are $|S|$ monochromatic pieces forming shape S.

11. For any numbers k and c there exists a dimension d such that each hypercube k^d colored with c colors contains a monochromatic combinatorial plane.

12. Prove Van der Waerden Theorem using Hales-Jewett Theorem.

13. Prove that $W(3, 2) = 9$.

14. Prove that for any number k among numbers $1, 2, 3, \ldots, n$ colored on 2 colors, there exists a monochromatic arithmetic sequence of length k, whose difference also has the same color.

15. Let $(s_1, s_2, s_3, \ldots)$ be an infinite sequence of ascending integers, such that $s_{i+1} - s_i < 1000$ for each $i \geq 1$. Prove that it contains arbitrarily large arithmetic sequence.

16. Prove that $R(3, 4) = 9$.

17. Prove that $R(4, 4) = 18$.

18. Prove that $R(3, k) \leq \frac{k(k+1)}{2}$.

19. Prove that $R(k, l) \leq \binom{k+l-2}{k-1}$.

20 (Schur)**.** Prove that for every integer c, there exists some N such that for every c-coloring of the set $\{1, 2, \ldots, N\}$, one of the color classes contains numbers x, y and z such that $x + y = z$.

21. Prove that for every $m \geq 1$, there is p_0 such that for any prime $p \geq p_0$, the congruence $x^m + y^m = z^m \mod p$ has a solution.

Łukasz Bożyk - Feel the Flow

Let $\mathcal{G} = (V, E)$ be a finite directed simple graph, i.e. $|V| < \infty$ and $E \subseteq V \times V \setminus \{(v, v) \mid v \in V\}$ and let $c\colon V \times V \to \mathbb{R}_{\geqslant 0}$ be a *capacity* function, i.e. if $(u, v) \in E$, then $c(u, v) = 0$. Throughout the following definitions and exercises we will consider a *flow network* $(\mathcal{G}, c)$.

We will use a convenient notation-simplifying convention: if $f\colon V \to \mathbb{R}$, $g\colon V \times V \to \mathbb{R}$ and $A, B \subseteq V$, $A \cap B = \varnothing$, then let

$$f(A) := \sum_{v \in A} f(v) \quad \text{and} \quad g(A, B) = \sum_{(u,v) \in A \times B} g(u, v).$$

Definition. Given $s, t \in V$, $s \neq t$, a function $\phi\colon V \times V \to \mathbb{R}_{\geqslant 0}$ is called an (s, t)-*flow* iff

$\forall v \in V \setminus \{s, t\}\ \phi(\{v\}, V \setminus \{v\}) = \phi(V \setminus \{v\}, \{v\})$,

$\forall (u, v) \in V \times V\ \phi(u, v) \leqslant c(u, v)$,

$\phi(\{s\}, V \setminus \{s\}) = \phi(V \setminus \{t\}, \{t\})$,

$\phi(V \setminus \{s\}, \{s\}) = \phi(\{t\}, V \setminus \{t\}) = 0$.

Special vertices s and t are called *source* and *sink* of the flow, respectively. The common value of both sides of the equality in

Definition. Given $s, t \in V$, $s \neq t$, a pair $(S, T) \in V \times V$ is called an (s, t)-*cut* iff $S \cup T = V$, $S \cap T = \varnothing$, $s \in S$, $t \in T$. The value $c(S, T)$ is called a *capacity* of the cut.

1.1. Prove that

1.2. Prove that a *minimal cut*, i.e. an (s, t)-cut of minimal capacity, always exists.

1.3. Prove[3] that a *maximal flow*, i.e. an (s, t)-flow of maximal value, always exists.

1.4. Prove that if (S, T) is an (s, t)-cut in $(\mathcal{G}, c)$, then for every (s, t)-flow ϕ holds

$$\phi(S, T) - \phi(T, S) = |\phi|.$$

1.5. Get familiar with the following theorem (make sure you know what it is about) and try to prove it (not using flows).

Theorem (Hall). We are given two finite sets: X, consisting of men, and Y, consisting of women, equipped with some (symmetric) familiarities (i.e. there are some male-female pairs who know each other). For $A \subset X$ define $N(A) \subset Y$ to be the set of all the women who know at least one man from A. Then it is possible for each man to marry a (different) woman who he knows iff for every $A \subset X$ the following inequality holds: $|N(A)| \geqslant |A|$.

[3] Solution may require basic analytical tools and a bit of knowledge about the topology of $\mathbb{R}^n$.

Theorem (Ford, Fulkerson). In a finite flow network $(\mathcal{G}, c)$ with source s and sink t the value of a maximal (s,t)-flow is equal to the capacity of a minimal (s,t)-cut:

$$\min\{c(S,T)\colon (S,T) \text{ is an } (s,t)\text{-cut}\} = \max\{|\phi|\colon \phi \text{ is an } (s,t)\text{-flow}\}.$$

Definition. In $(\mathcal{G}, c)$ the capacity function c is called *integral* iff $c\colon V \times V \to \mathbb{Z}_{\geqslant 0}$. Similarly, an (s,t)-flow ϕ is called *integral* iff $\phi\colon V \times V \to \mathbb{Z}_{\geqslant 0}$.

2.1. Prove that if ϕ is a maximal (s,t)-flow and (S,T) is a minimal (s,t)-cut in $(\mathcal{G}, c)$, then

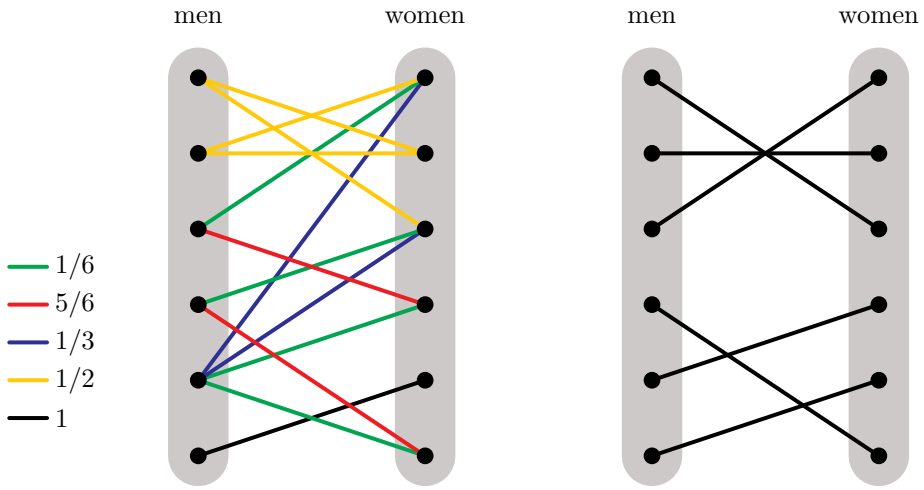
$$\phi(S,T) = c(S,T),$$

i.e. for every $(u,v) \in S \times T$ we have $\phi(u,v) = c(u,v)$.

2.2. Prove that in a finite flow network with integral capacity always exists a maximal (s,t)-flow which is integral.

2.3. On a *Convex Monogamy Planet* live n men and n women, and there are some male-female relations there. Each such relation is a *marriage to some extent*, i.e. there exists a number $t \in [0,1]$ (*marital weight*) indicating how much of a marriage the relation is.

Each person on the planet has altogether *exactly one* spouse, so for each the sum of marital weights equals 1 (e.g. one can have two *half-wives* or four *quarter-husbands*; a full example is shown in the picture below).



After elections the new emperor said 'This is just wrong.' and decreed a (good) change in the marital law. Prove that it is possible to remarry the planet's citizens so that each man is married to exactly one women (with whom he had positive marital weight before the reform).

2.4. On a *Discrete k-Polygamy Island* live men and women, and there are some male-female familiarities there (a structure of a bipartite graph). Each man can have at most k wives, whereas each woman can have at most one husband. Of course everyone has to know all of their spouses.

We will call a *perfect matching* a situation when each man has exactly k wives. Prove the following generalisation of Hall's theorem: a perfect matching exists iff for every subset A of the set of all men holds

$$|N(A)| \geqslant k \cdot |A|,$$

where $N(A)$ is the set of all women who know at least one man from A.

DEFINITION. Given a flow network $(\mathcal{G}, c)$ and a function $f\colon V \to \mathbb{R}$, a function $\psi\colon V \times V \to \mathbb{R}$ is called an *f-flow* iff

$$\forall v \in V\ \psi(\{v\}, V \setminus \{v\}) = f(v),$$

$$\forall (u,v) \in V \times V\ \psi(u,v) \leqslant c(u,v),$$

$$\forall (u,v) \in V \times V\ \psi(u,v) = -\psi(v,u).$$

THEOREM (F-F/f-F). For a finite flow network $(\mathcal{G}, c)$ a function $f\colon V \to \mathbb{R}$ is given. An f-flow in $(\mathcal{G}, c)$ exists iff for every $F \subseteq V$

$$-c(F', F) \leqslant f(F) \leqslant c(F, F'),$$

where $F' := V \setminus F$.

3.1. A network $(\mathcal{G}, c)$ with fixed $s, t \in V$, $s \neq t$ and a number $y \in \mathbb{R}_{\geqslant 0}$ are given. Prove[4] that an (s,t)-flow of value y exists iff there exists an f-flow with f defined as: $f(s) = -f(t) = y$ and $f(v) = 0$ for $v \notin \{s, t\}$.

3.2. Derive the classical version of Ford-Fulkerson theorem from the version for f-flows.

3.3. Prove that if c is integral, $f\colon V \to \mathbb{Z}$ and in $(\mathcal{G}, c)$ there exist an f-flow, then in $(\mathcal{G}, c)$ exists an integral f-flow.

From now on let us consider graphs $\mathcal{G} = (V, E)$ with $|V| < \infty$, $V = X \cup Y$, $X \cap Y = \varnothing$, $E \subseteq X \times Y$, i.e. finite bipartite graphs with all edges directed (for our convenience) from X to Y. For sets $A \subseteq X$, $B \subseteq Y$, define partial complements: $A' := X \setminus A$, $B' := Y \setminus B$.

DEFINITION. We will say that in $\mathcal{G}$ exists a *k-matching from X to Y* iff there exist k-element sets $\{x_i\}_{i=1}^k \subseteq X$, $\{y_i\}_{i=1}^k \subseteq Y$ such that $(x_i, y_i) \in E$ for each $i \in [1,k] \cap \mathbb{Z}$.

THEOREM (ON BIPARTITE k-MATCHINGS, BMT). In $\mathcal{G}$ a k-matching from X to Y exists iff for every pair $A \subseteq X$, $B \subseteq Y$ holds

$$|(A' \times B') \cap E| + |A| + |B| \geqslant k.$$

3.4. Derive Hall's theorem and König's theorem from BMT.

3.5. Derive BMT from König's theorem.

3.6. We'll say that a finite directed simple graph $\mathcal{G}$ with $|V| = n$ has a *full cycle decomposition* iff there exist an n-element set $\{(x_i, y_i)\}_{i=1}^n \subseteq E$ such that

$$\bigcup_{i=1}^{n} \{x_i\} = \bigcup_{i=1}^{n} \{y_i\} = V.$$

Prove that $\mathcal{G}$ has a full cycle decomposition iff for every pair $(A, B) \subseteq V \times V$ holds

$$|(A \times B) \cap E| + |V| \geqslant |A| + |B|.$$

State and prove analogous theorem for undirected graphs.

[4] Seems quite obvious but turns out to be surprisingly tricky!

Ján Gašper - Logic

Introduction

In this series of talks we'll talk about logic and processor. As we know, both of them work with Trues and Falses; ones and zeros. We'll find out exactly how and we'll build some circuits in dedicated software as well as in Minecraft.

1 Logic and binary

Table of some symmetric logic operations:

A	B	AND	NAND	OR	NOR	IFF (XNOR)	XOR
0	0	0	1	0	1	1	0
0	1	0	1	1	0	0	1
1	0	0	1	1	0	0	1
1	1	1	0	1	0	1	0

Note that AND and NAND, OR and NOR, XOR and XNOR has inverted logic values. The letter 'N' at the beginning marks 'not-'. XOR is abbreviation of 'exclusive or'. It is often used as conditional negation where A is input and B is condition. There will be also NOT gates with one input, returning the opposite value and Buffer, passing with one input the same value further.

Fact 1.1. *Every natural number can be written as sum of different powers of 2 in a unique way.*

Problem 1.1. *Write numbers 106, 89 and 195 in binary*

Solution 1.1.

$$106 = 64 + 32 + 8 + 2 = 2^6 + 2^5 + 2^3 + 2^1$$
$$89 = 64 + 16 + 8 + 1 = 2^6 + 2^4 + 2^3 + 2^0$$
$$195 = 128 + 64 + 2 + 1 = 2^7 + 2^6 + 2^1 + 2^0$$

Adding two integers

Problem 1.2. *Add numbers 106 and 89 in binary*

Solution 1.2.

128	64	32	16	8	4	2	1
1	*1*	*1*	*1*	*0*	*0*	*0*	
	1	1	0	1	0	1	0
	1	0	1	1	0	0	1
1	1	0	0	0	0	1	1

Which is exactly 195, as we can see from Problem 1.1

Multiplying two integers

Problem 1.3. *Muliply* 11×5 *in binary.*

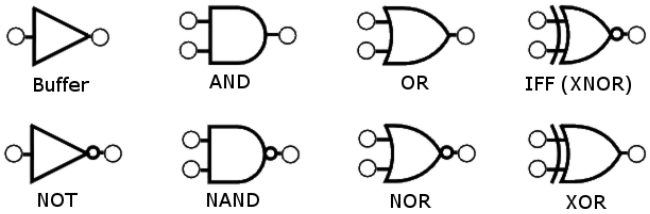
Solution 1.3.

32	16	8	4	2	1
		1	0	1	1
		×	1	0	1
		1	0	1	1
1	0	1	1		
1	1	0	1	1	1

which is 55.

2 Modeling circuits

For circuits we'll use logic gates. They do the same as logic operations, but with voltage. There is a convention that high voltage means the same as true or one. Low voltage means the same as false or zero. (High voltage is usually 5V or 12V) We'll use these symbols for logic gates:



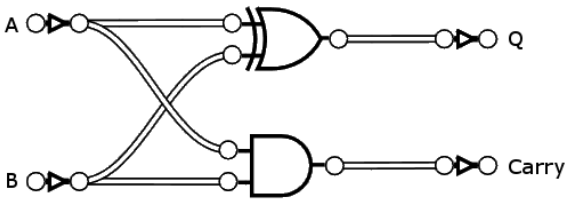
Adding circuit

1-bit adder circuit

Problem 2.1. *Make a truth table and a circuit that adds two 1-bit integers*

Solution 2.1.

A	B	Q	Carry
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1



Please note that output is marked as 'Q'. This is a convention, that prevent us from misreading 'O' for 'output' as zero. The next bit should be carried in next column, so it is marked as 'Carry'.

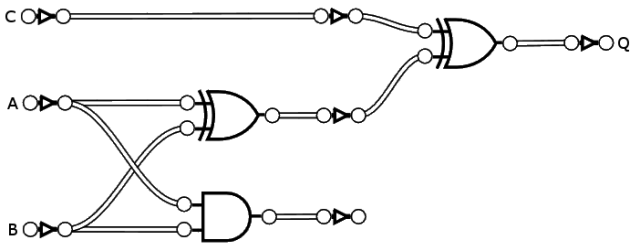
Adder unit

In order to add bigger numbers, we need to build a circuit with three inputs (A,B and Carry in) and two outputs (Q and Carry out).

Problem 2.2. *Make a truth table and circuit for adder unit. You might use result of Problem 1.1.*

Solution 2.2.

C	A	B	Q	C
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1



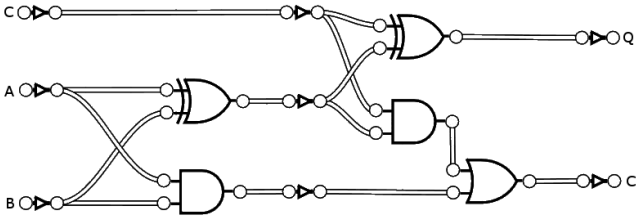
Half adder

Now we'll use the circuit from last problem as it solves the first half of the table. For second half notice that Q is inverted. That's conditional negation, so we'll use XOR gate with condition C. Now we have:

But carry out is still not finished. For that we observe that it is true in this cases:

- A AND B are both true, no matter what C is, OR
- C is true and A XOR B is true

Since we already have A XOR B in our circuit, we might save some gates. The finished circuit looks like this:



Full adder

8-bit adder

Let's say we want to build a 8-bit computer. For that we will need a circuit that adds 8-bit integers and outputs their 8-bit sum. For that we'll just stacks eight adding units from previous problem. We can check that our circuit works correctly by adding $106 + 89$.

But as it outputs only 8 bits, we throw away $2^8 = 256$, that could occur. But this is actually a feature. Instead of working with all natural numbers, we are operating in a world "modulo 256". This helps us build negative numbers. From algebra we know that we can choose also negative numbers to be members of a modulo structure. In computers are negative numbers considered those with leading digit one. (So the leading digit is a 'sign' of the number.) So instead of using numbers 0 to 255 we have -128 to 127. (Unless we choose to use 'unsigned' integers.) We can easily make a negative number from positive and vice versa. All that is needed is to invert all the digits and add one. It's called 'twos complement'. We can instantly see, that $-0 = 0$.

Problem 2.3. *Make -89 as 8-bit integer*

Solution 2.3.

From Problem 1.1 we know

$$89 = 01011001_2$$

, inverted

$$10100110_2$$

and added to one

$$10100110_2 + 1_2 = 10100111_2$$

Double-check. If we add $89 + (-89)$, we should get 0:

1	*1*	*1*	*1*	*1*	*1*	*1*	
0	1	0	1	1	0	0	1
1	0	1	0	0	1	1	1
0	0	0	0	0	0	0	0

And as we can see, the the one we added in twos complement is just carried all the way and overflowed.

Problem 2.4. *Compute* $106 - 89$ *using twos complement*

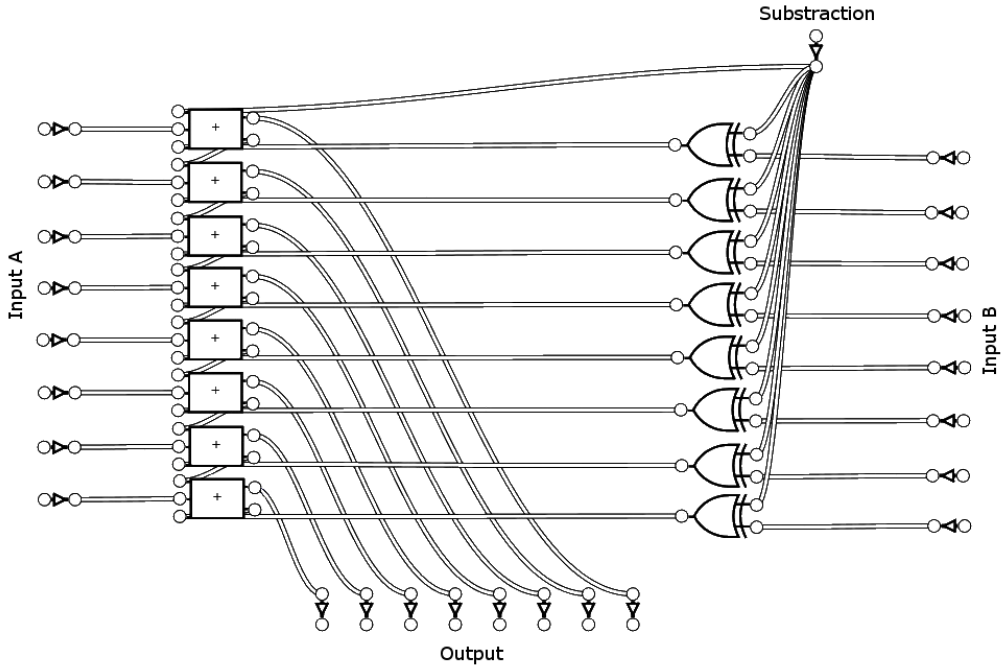
Solution 2.4.

1	*1*	*0*	*1*	*1*	*1*	*0*	
0	1	1	0	1	0	1	0
1	0	1	0	0	1	1	1
0	0	0	1	0	0	0	1

which is exactly 17 as expected.

ALU

Arithmetic-logic unit, or 'ALU' is that part of computer where all computations are done. We want our ALU to be able to add numbers, substract numbers and compare them. Our built circuit is already able to add numbers. For substraction we need to build an number inverter that we can turn on or off. For that we'll use XOR gates. After that we should add one to it to accomplish twos complement. For that we use carry in for ones, which is still unused.



ALU

Problem 2.5. *Find the best way to compare two numbers.*

Solution 2.5.

To compare two numbers we just look at the sign of their difference.

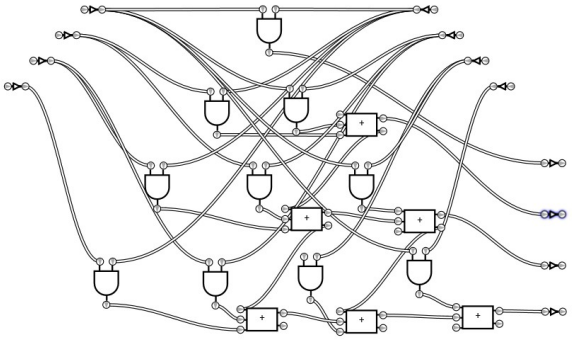
Multiplying circuit

This circuit is not usually used, because it is "too expansive". As you can see, only 4-bit integer multiplication circuit is total mess. However, problem of multiplication is still open problem in informatics. (The most effective way of multiplication is by now solved in $n \log n$ time where n is length of the bigger number.) The most common solution is iterated shift-add algorithm.

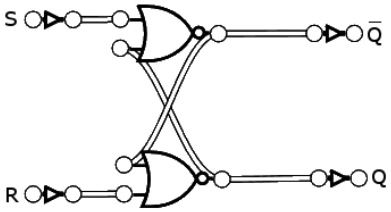
3 Memory circuits

S-R latch

The simplest memory circuit is S-R latch. After the set signal is input, circuit outputs one, or similary reset signal cause output zero until set is input again.

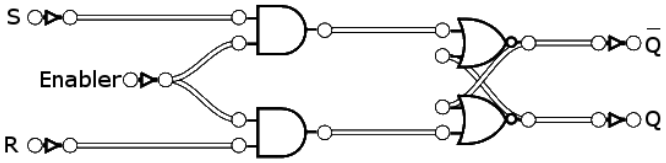


Multiplication mess



The idea of S-R latch is found in all static memory circuits. This circuit besides output Q outputs also negated output $\overline{Q}$, which can be useful in some situations.

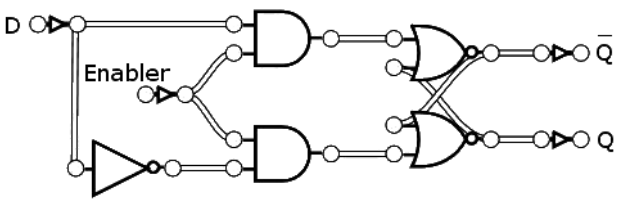
It's useful to have input enabler in latch, so you won't accidentally input a value. We can do this by adding AND gate in front of inputs. Now we have S-R latch with enabler.



The dark side of it is that when we input both of S and R, latch will output zero at both Q and $\overline{Q}$. After releasing them it will output random number. We'll see how this glitch can be avoided.

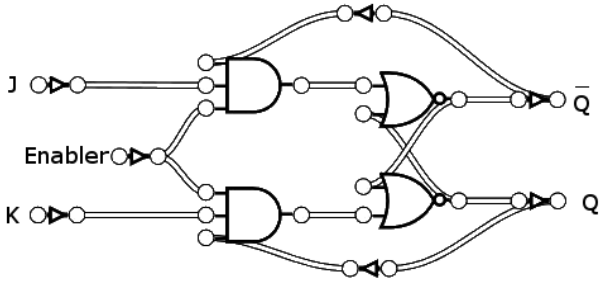
D latch

The D latch is a simple modification of S-R latch. Instead of setting or resetting, it will output the last data on the input. (That's why it is called D latch - d for data.) In order to use it as a memory block we have to add also input enabling signal.



J-K flip-flop

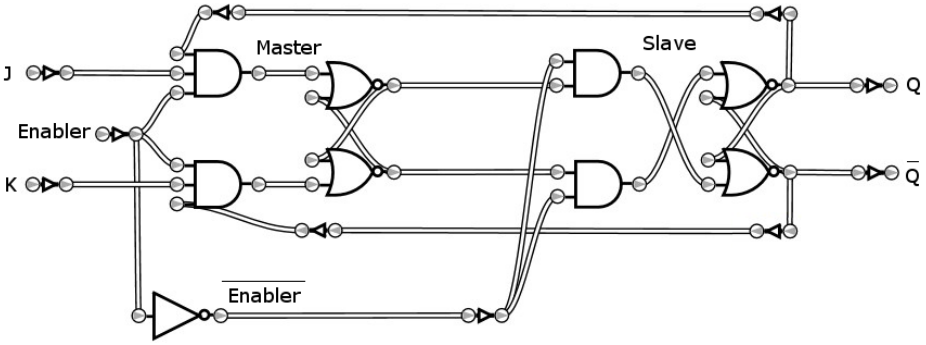
J-K flip-flop (or J-K latch) is another approach to problem of random number output of S-R latch. Instead of outputting random value, after pressing both J and K inputs the output will toggle. We can achieve this by allowing only one input signal come in. For this we'll use three input and gate.



This solution works, but only if you have sufficiently short enabler pulse. If you use longer enabler pulse, J-K flip-flop may actually toggling several times (and thus output random value).

J-K master-slave flip-flop

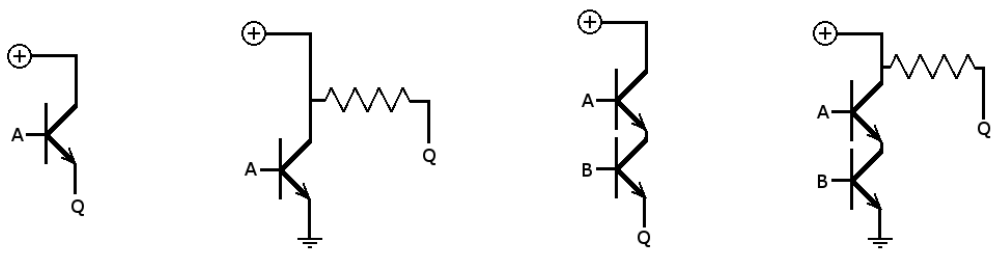
This solution also uses the idea of allowing only one input pass, but in a different way. This latch consists of two J-K latches, one of the top of the other. They must not be both active at the same time, so multiple toggling is prevented. For this, we use enabler signal to one and inverted enabler to the second. The one into which comes input is called master (gives orders to the other), and the other is called slave (obeys master).

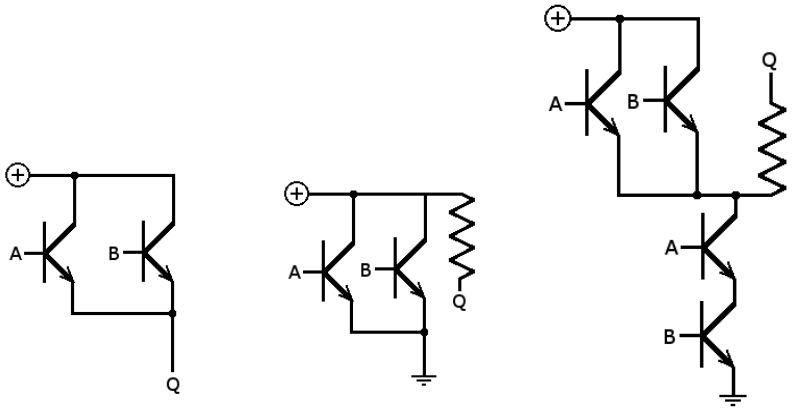


Appendix

Logic gates from transistors

Problem 3.1. *Decide which figure belongs to which logic gate*





Solution 3.1.

Buffer, NOT, AND, NAND, OR, NOR, XOR in this order.

Buffers and ROMs

Buffer is the simplest logic gate. All it does is passing value it has recieved. That might be good in two cases:

- You need signal to get to full power again
- You want signal to pass only one direction

But it has also a con - it delays signal a little bit.

Special case is tri-state buffer. It remembers last value you entered (D-latch) and releases it when you activate output using AND gates. This is very useful when building registers and communicating with the bus.

Gábor Damásdi - Combinatorial Geometry

Reminders

- Three or more lines are said to be **concurrent** if they intersect at a single point.
- Three or more points are said to be **collinear** if they lie on a straight line.

Problems

(a) There are n houses along a road. Where should we build a bus stop, if we want to minimize the sum of the distances between the bus stop and the houses?

(b) Let S be a set of $2n$ distinct points in the plane. A **halving line** is a line that splits the points into two equal-sized subsets.

  (a) Show that there exists a halving line for S.

  (b) Show that you can choose the direction of the halving line almost freely.

(c) Given n lines in the plane in general position (no two of them are parallel, and no three of them are concurrent). How many regions do these lines define in the plane?

(d) Given $2n+3$ points in the plane such that no three are collinear and no four lies on a circle. Show that you can find three among the points such that their circumscribed circle contains exactly n of the points.

(e) Let S be a finite set of at least two points in the plane. Assume that no three points of S are collinear. A **windmill** is a process that starts with a line l going through a single point $P \in S$. The line rotates clockwise about the pivot P until the first time that the line meets some other point belonging to S. This point, Q, takes over as the new pivot, and the line now rotates clockwise about Q, until it next meets a point of S. This process continues indefinitely.

- Is it true that each point becomes a pivot during the process?
- Is it true that the line always reaches its original position again?
- What can we say about the number of points in the two halfplanes through the process?

(f) Let S be a finite set of at least two points in the plane. Assume that no three points of S are collinear. Show that we can choose a point P in S and a line l going through P such that the resulting windmill uses each point of S as a pivot infinitely many times.

(52. IMO 2011 Problem 2)

(g) Given n lines in the plane, such that no three of them are concurrent. We want to color the intersections such that if two intersections are neighbours along a line, then they have different colors. Show that such coloring always exists using three colors.

(h) Given n lines in the plane, such that no three of them are concurrent. We want to write a number on each intersections such that the numbering is strictly monotone along every line. Show that such numbering always exists.

(i) Let S be a set of $4n$ points in the plane. Assume that no three points of S are collinear. Is it true that there always exists two lines such that they split the points into four equal-sized subsets?

(j) Let S be a set of $2n$ red points and Q be a set of $2n$ blue points in the plane. Assume that no three points of $S \cup Q$ are collinear. Is it true that there always exists a line that is a halving line for booth S and Q?

(k) Let S be a set of $7n$ points in the plane. Assume that no three points of S are collinear. Is it true that there always exists three lines such that they split the points into seven equal-sized subsets?

(l) Let S be a set of $6n$ points in the plane. Assume that no three points of S are collinear. Is it true that there always exists three concurrent lines such that they split the points into six equal-sized subsets?

(m) Let S be a set of $8n$ points in the plane. Assume that no three points of S are collinear. Is it true that there always exists four concurrent lines such that they split the points into eight equal-sized subsets?

(n) Let S be a set of $4n$ points in the plane. Assume that no three points of S are collinear. Is it true that there always exists two **orthogonal** lines such that they split the points into four equal-sized subsets?

(o) Let S be a set of $3n$ points. Prove that there is a point p in the plane, such that any half plane with p on its boundary contains at least n points from S.

If a point p satisfies this condition, it is called a **center point** of set S.

Show that n is the best possible we can achieve, so it is not always true that there is a point p in the plane, such that any half plane with p on its boundary contains at least $n+1$ points from S.

(p) Geoff has placed 64 points on the plane according to a 8×8 grid. Then he wants to divide the plane into regions using lines such that no region contains more than one point. How many lines does he need?

(q) Given n red and n blue points in the plane. Show that we can draw n segments such that each segment has one red and one blue endpoint, and no two segments cross or share an endpoint.

(r) Given $3n$ points in the plane in general position. Show that we can group the points into groups of three, such that the triangles determined by the groups are non-intersecting.

(s) Given $3n$ points in the plane in general position. Show that we can group the points into groups of three, such that the intersection of the triangles determined by the groups is nonempty.

(t) Given n points in the plane in general position. They determine $\binom{n}{3}$ triangles. Show that there is a point p in the plane that lies in at least $n^3/27 + O(n^2)$ of the triangles.

Combinatorial geometry from IMO.

(a) There are $n \geq 2$ line segments in the plane such that every two segments cross, and no three segments meet at a point. Geoff have to choose and endpoint of each segment and place a frog on it, facing the other endpoint. Then he will clap his hands $n-1$ times. Every time he claps, each frog will immediately jump forward to the next intersection point on the segment. Frogs never change the direction of their jumps. Geoff wishes to place the frogs in such a way that no two of them will ever occupy the same intersection point at the same time.

- Prove that Geoff can always fulfil his wish if n is odd.
- Prove the Geoff can never fulfil his wish if n is even.

(57. IMO 2016 Problem 6)

(b) A configuration of 4027 points in the plane is called Colombian if it consists of 2013 red points and 2014 blue points, and no three of the points of the configuration are collinear. By drawing some lines, the plane is divided into several regions. An arrangement of lines is good for a Colombian configuration if the following two conditions are satisfied:

- no line passes through any point of the configuration;
- no region contains points of both colours.

Find the least value of k such that for any Colombian configuration of 4027 points, there is a good arrangement of k lines.

(54. IMO 2013 Problem 2)

(c) A set of lines in the plane is in general position if no two are parallel and no three pass through the same point. A set of lines in general position cuts the plane into regions, some of which have finite area; we call these finite regions. Prove that for all suficiently large n, in any set of n lines in general position it is possible to colour at least $\sqrt{n}$ of the lines blue in such a way that none of its finite regions has a completely blue boundary.

Note: Results with $\sqrt{n}$ replaced by $c\sqrt{n}$ will be awarded points depending on the value of the constant c.

(55. IMO 2014 Problem 6)

(d) We say that a finite set $\mathcal{S}$ of points in the plane is balanced if, for any two different points A and B in $\mathcal{S}$, there is a point C in $\mathcal{S}$ such that $AC = BC$. We say that $\mathcal{S}$ is centre-free if for any three different points A, B and C in $\mathcal{S}$, there is no points P in $\mathcal{S}$ such that $PA = PB = PC$.

(a) Show that for all integers $n \geq 3$, there exists a balanced set consisting of n points.

(b) Determine all integers $n \geq 3$ for which there exists a balanced centre-free set consisting of n points.

(56. IMO 2015 Problem 1)

Kada Williams - Symmetric Polymnomials

Symmetric sums:

Fix $x_1, x_2, \ldots, x_n$. We denote $\sigma_k = \sum_{i_1<i_2<\ldots<i_k} x_{i_1}x_{i_2}\ldots x_{i_k}$ and $S_\ell = \sum_{i=1}^{n} x_i^\ell$.

1. Let $n = 3$. Express the sums $S_1, S_2, \ldots, S_5$ using $p = \sigma_1$, $q = \sigma_2$ and $r = \sigma_3$.

2. The roots of $x^3 + px + q$ are $x_1, x_2, x_3 \in \mathbb{R}$. Prove that

$$x_1^5 + x_2^5 + x_3^5 = 5pq.$$

3. Solve the system

$$\begin{aligned} x + y + z &= 1 \\ x^3 + y^3 + z^3 &= 91 \\ x^5 + y^5 + z^5 &= 4651 \end{aligned}$$

4. [Newton-Girard formulae] Prove that for $m < n$,

$$S_m - \sigma_1 H_{m-1} + \sigma_2 H_{m-2} \mp \ldots + (-1)^m \sigma_m m = U_0 - (U_0 + U_1) + (U_1 + U_2) \mp \ldots + (-1)^m U_m = 0.$$

(I won't tell you what U_i means.) Using different ideas, find a similar formula for $m \geq n$.

With some concrete computational experience, let's prove the **Fundamental Theorem of Symmetric Polynomials**. Namely, if R is a commutative ring, then every symmetric polynomial $f(x_1, x_2, \ldots, x_n)$ with coefficients from R can be expressed as a polynomial in $\sigma_1, \sigma_2, \ldots, \sigma_n$ with coefficients from R. [I'll explain what this means.]

One proof is to use Gauss's algorithm and lexicographical ordering. Another proof is by induction on the number of variables. Are they the same?

Root $\Leftrightarrow$ polynomial:

5. Find a polynomial $p \in \mathbb{Z}[x]$ with $p(\sqrt{5} + \sqrt{7}) = 0$. For bonus points, find $q \in \mathbb{Z}[x]$ with $q(\sqrt{5} + \sqrt[3]{7}) = 0$.

6. Let $\phi = \frac{1+\sqrt{5}}{2}$. How would you determine the parity of $\lfloor \phi^n \rfloor$?

7. [Komal A.465.] Prove that for all positive integers n, $\left\lfloor (\sqrt[3]{28} - 3)^{-n} \right\rfloor$ is not divisible by 6.

8.* [Komal A.502.] Prove that for arbitrary complex numbers $w_1, w_2, \ldots, w_n$ there exists a positive integer $k \leq 2n+1$ for which $\mathrm{Re}(w_1^k + \ldots + w_n^k) \geq 0$.

Combining roots $\Leftrightarrow$ combining polynomials:

9. Let a, b be two different roots of $x^4 - x^3 - 1$. Prove that ab is a root of $x^6 + x^4 + x^3 - x^2 - 1$.

10. Every algebraic $\alpha \in \mathbb{C}$ can be assigned a minimal polynomial $f \in \mathbb{Z}[x]$. Prove that $g(\alpha) = 0$ ($g \in \mathbb{Z}[x]$) implies $f|g$.

11. Prove that the algebraic numbers form a ring. Prove that the *algebraic integers* form a ring.

12. Prove that the $\sqrt{}$'s of the squarefree positive integers are linearly independent over $\mathbb{Q}$.

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Bonus problems:

A) Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be pairwise distinct real numbers, and consider the $n \times n$ "addition table", where the entry in the i-th row and j-th column is $a_i + b_j$. Prove that if all the column products are equal, then all the row products are equal.

B) Let $p(x) = x^3 - x^2 - 2x + 1$. Prove that $a - ab = 1$ for some roots a, b of $p(x)$.

C) Let $x + y + z = 0$ and $xyz = 2$. Maximize $S = x^3y + y^3z + z^3x$.

D) [IMO Shortlist] Let S be the sum of positive integers a, b, c, d, e, f. Prove that if S divides $abc - def$ and $(ab + bc + ca) - (de + ef + fd)$, then S cannot be prime.

E) Suppose that every complex root of monic $f \in \mathbb{Z}[x]$ is on $|z| = 1$. Prove that $f(x)|(x^n - 1)^k$ for some $k, n \in \mathbb{N}$.

Mateusz Kobak - Intro to Galois Theory

1 Chapter One - Field Theory

Definition 1.1. A field is a set F together with two operations $+$ and $\cdot$ such that:

(a) $(F, +)$ is an abelian group,

(b) $(F, \cdot)$ is an abelian group,

(c) the distributive law holds.

Note that a field is actually a commutative ring, so homomorphism of fields is simply a homomorphism of rings.

Characteristic of a field

Definition 1.2. Characteristic of a field is the smallest number of times one must use the element 1 in a sum to obtain element 0. If there is no option to obtain 0, we say that a field has characteristic 0.

Corollary. *Characteristic of a field is either* 0 *or a prime.*

Corollary. *A finite field has* p^n *elements for a prime number* p *and positive integer* n*.*

Definition 1.3. Fields $\mathbb{F}_2, \mathbb{F}_3, \ldots, \mathbb{Q}$ are called simple fields, as they do not contain any proper subfield.

Corollary. *Every field contains a copy of exactly one simple field.*

Extensions

Definition 1.4. An extension E of field F is a field that contains F as a subfield.

The intersection of subfields is again a subfield. Let F be a subfield of E and S a subset of E. Than the intersection of all subfields of E containing F and S must be the smallest subfield containing F and S. We call it the **subfield generated by** F **and** S **(or generated over** F **by** S**)** and we denote it $F(S)$. When $S = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ we call $F[\alpha_1, \alpha_2, \ldots \alpha_n]$ a set of polynomials in the α_i's with coeficients in F and than $F(\alpha_1, \alpha_2, \ldots \alpha_n)$ is a set of quotients of such polynomials.

An extension E of F is said to be simple if $E = F(\alpha)$ for some $\alpha \in E$. For example $\mathbb{Q}[i]$ and $\mathbb{Q}(\pi)$ are simple fields.

Proposition 1.1. *For a monic irreducible polynomial* $f(x)$ *of degree* m *in* $F[x]$*,*

$$E = F[x]/\left(f(x)\right)$$

is a field extension of degree m *over* F*.*

Definition 1.5. By $[E : F]$ we denote the dimension of E regarded as an F-vector space. We call it the degree of E over F.

Proposition 1.2 (Multiplicativity of degrees)**.** *Consider fields* $L \supset E \supset F$*. Than* L/F *is of finite degree if and only if both* L/E *and* E/F *are of finite degree, in which case*

$$[L : F] = [L : E] \cdot [E : F].$$

If E and E' are extension fields of F than an F**-homomorphism** $\varphi : E \to E'$ is such a homomorphism that $\varphi(c) = c$ for all $c \in F$.

Definition 1.6. Let f be a monic irreducible polynomial in $F[x]$. A pair (E, α) consisting of an axtension E of F and $\alpha \in E$ is called a **stem field** for f if $E = F[\alpha]$ and $f(\alpha) = 0$.

Proposition 1.3. *Stem field* (E, α) *for* $f \in F[x]$ *is* F*-isomorphic to* $E = F[x]/\left(f(x)\right)$*.*

Algebraic and transcendental elements

For an element $\alpha \in E \supset F$ we say that α is **algebraic** over F if there exists $f \in F[x]$ such that $f(\alpha) = 0$. Otherwise we call α **transcendental** over F.

Definition 1.7. For an element $\alpha \in E \supset F$ we say that $f \in F[x]$ is the **minimum polynomial** of α over F if $f(\alpha) = 0$ and f is monic and irreducible.

A field extension is called algebraic if all elements of E are algebraic over F, otherwise it is called to be transcendental over F.

Theorem 1.4. *A field extension E/F is finite if and only if E is algebraic and finitely generated over F.*

Definition 1.8. (a) A field Ω is said to be algebraically closed if every polynomial in $\Omega[x]$ splits in Ω.

(b) A field Ω is an algebraic closure of F when it is algebraicly closed and algebraic over F.

Exercises

(a) Deterimne $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}]$

(b) Prove that every finite subgroup of $F^\times$ is cyclic.

(c) Let $f(x)$ be irreducible polynomial over F of degree n, and let E be an extension of F with $[E : F] = m$. Show that if $\text{nwd}(m, n) = 1$, then f is irreducible over E.

(d) Construct a field with p^n elements.

2 Chapter two - Splitting Fields

Definition 2.1. Let M, L be fields. A homomorphism $\varphi : M \to L$ is a map, such that:

(a) $\varphi(0) = 0$

(b) $\varphi(1) = 1$

(c) $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$

(d) $\varphi(a + b) = \varphi(a) + \varphi(b)$

Every homomorphism of fields is an injection.

Definition 2.2. Let M, L be fields. An isomorphism $\phi : M \to L$ is a map, such that, there exists homomoprhism $\psi : L \to M$ such that:

(a) For every $m \in M$, $\psi(\phi(m)) = m$

(b) For every $l \in L$, $\phi(\psi(l)) = l$.

Automorphism is an isomorphism from a field K to itself.

Theorem 2.1. *Let $F[\alpha]$ be a simple algebraic field extension, and let Ω be a second field containing F. Let $f(x)$ be the minimum polynomial of α over F. For every F-homomorphism $\varphi : F[\alpha] \to \Omega$, $\varphi(\alpha)$ is a root of $f(x)$ in Ω and the map $\varphi \mapsto \varphi(\alpha)$ defines a one-to-one correspondence*

$$\{F\text{-homorphisms } \varphi : F[\alpha] \to \Omega\} \leftrightarrow \{\text{roots of } f \text{ in } \Omega\}.$$

In particular, the number of such maps is the number of distinct roots of f in Ω.

Proposition 2.2. *Let $F(\alpha)$ be a simple, algebraic extension of F, and let $\varphi_0 : F \to \Omega$ be a homomorphism of F into a second field Ω. Let $f(x)$ be a minimum polynomial of α over F. Then the map $\varphi \mapsto \varphi(\alpha)$ defines a one-to-one correspondence*

$$\{\text{extensions } \varphi : F[\alpha] \to \Omega \text{ of } \varphi_0\} \leftrightarrow \{\text{roots of } \varphi_0(f) \text{ in } \Omega\}.$$

In particular, the number of such maps is the number of distinct roots of f in Ω.

Splitting Field

Definition 2.3. If $f \in F[x]$ splits in $E \supset F$ and in addition

$$E = F[\alpha_1, \alpha_2, \ldots, \alpha_m],$$

where $\alpha_1, \alpha_2, \ldots, \alpha_m \in E$ are all roots of f, than E is called a **splitting field** for f.

Proposition 2.3. *(Homoework xD) Every polynomial $f \in F[x]$ has a splitting field E_f and morover*

$$[E_f : F] \leqslant (degf)!$$

Proposition 2.4. *Let $f(x) \in F[x]$. Let E be a splitting field of f over F and let Ω be another field containing F in which f splits.*

(a) There exists an F-homomorphism $\varphi : E \to \Omega$; the number of such homomorphisms is at most $[E : F]$ and the equality holds if f has distinct roots.

(b) If E and Ω are both splitting fields for f, then they are F-isomorphic.

Corollary. *Let E and L be extension fields of F, with E finite over F.*

(a) The number of F-homomorphisms $E \to L$ is at most $[E : F]$.

(b) There exists a finite extension Ω/L and an F-homomorphism $E \to \Omega$.

Multiple roots

Distinct irreducible polynomials in $F[x]$ do not acquire a common root in any extension of F.

Let polynomial f split in E and $f = \prod(x - \alpha_i)^{m_i}$. If $m_i = 1$ then α_i is called a simple root and multiple root otherwise.

Proposition 2.5. *For a nonconstant irreducible polynomial f in $F[x]$ the following statements are equivalent:*

(a) f has a multiple root;

(b) $\mathrm{nwd}(f, f') \neq 1$*;*

(c) F has characteristic $p \neq 0$ and f is a polynomial in x^p;

(d) all roots of f are multiple;

Definition 2.4. A polynomial (not necceseraly irreducible) is said to be **separable** if it has only simple roots.

Exercises

(a) Let F be a field with characteristic $\neq 2$.

(a) Let E be quadratic extension of F (i.e. $[E : F] = 2$). Show that

$$S(E) = \{a \in F^\times \mid a\text{is a square in } E\}$$

is a subgroup of F.

(b) Let E and E' be quadratc extensions of F. Prove that E and E' are F-isomorphic if and only if $S(E) = S(E')$.

(c) Let p be a prime. Show that there is exactly one field with p^2 elements, up to isomorphism. Can it be generalised to fields with p^n elements?

(b) Let E be a splitting field of polynomial $x^5 - 2$ over $\mathbb{Q}$. Find $[E : Q]$.

3 Chapter three - Galois Theory

Group of automorphisms

We denote a group of F-automorphisms of $E \supset F$ as $\mathrm{Aut}(E/F)$.

Proposition 3.1. *If E is a splitting field of a separable polynomial $f \in F[x]$, then $Aut(E/F)$ has order $[E : F]$.*

For every group G of automorphisms of a field E we set

$$E^G = \mathrm{Inv}(G) = \{\alpha \in E \mid \sigma\alpha = \alpha, \forall \sigma \in G\}.$$

Ita appears to be a subfield of E, called the fixed field of G.

Theorem 3.2. *Let G be a finite group of automorphisms of a field E, and let $F = E^G$. Then*

$$[E : F] \leqslant |G|.$$

Corollary. *For any finite group G of automorphisms of a field E,*

$$G = Aut(E/E^G).$$

Definition 3.1. An algebraic extension E/F is called separable if the minimum polynomial of every element in E is separable; otherwise it is inseparable.

Definition 3.2. An algebraic extension E/F is called normal if the minimum polynomial of every element of E splits in E.

Definition 3.3. An finite algebraic extension E/F is said to be Galois if F is the fixed field of the group of F-automorphisms of E. This group is then called the Galois group of E over F, and is denoted by $\mathrm{Gal}(E/F)$.

Theorem 3.3. *For any extension E/F, the following statements are equivalent:*

(a) E is the splitting field for some $f \in F[x]$.

(b) $F = E^G$ for some finite group G of automorphisms of E.

(c) E/F is normal, separable and of finite degree.

(d) E/F is Galois.

Corollary. *Let $E \supset M \supset F$. If E is Galois over F, then it is also Galois over M.*

Fundamental theorem of Galois theory

Theorem 3.4 (Fundamental theorem of Galois theory)**.** *Let E be a Galois extension of F, and let $G = Gal(E/F)$. The maps $H \mapsto E^H$ and $M \mapsto Gal(E/M)$ are inverse bijections between the set of subgroups of G and the set of intermediate fields between E and F:*

$$\{subgroups\ of\ G\} \leftrightarrow \{intermediate\ fields\ E \supset M \supset F\}.$$

Morover,

(a) the correspondence is inclusion-reversing: $H_1 \supset H_2 \Leftrightarrow E^{H_1} \subset E^{H_2}$

(b) indexes equal degrees: $(H_1 : H_2) = [E^{H_2} : E^{H_1}]$

(c) $\sigma H \sigma^{-1} \leftrightarrow \sigma M$, i.e. $E^{\sigma H \sigma^{-1}} = \sigma(E^H)$ and $Gal(E/\sigma M) = \sigma\, Gal(E/M)\sigma^{-1}$

(d) H is normal in $G \Leftrightarrow E^H$ is normal (hence Galois) over F, in which case

$$Gal(E^H/F) \simeq G/H.$$

Qualifying Quiz

Qualifying Quiz - Problems

Problem 1. Determine all non-zero integers a, for which there exists an integer b such that: $ab|a^2+b^2-a$.

Problem 2. Consider a positive integer n and the function $f:\mathbb{N}\to\mathbb{N}$ by

$$f(x)=\begin{cases}\frac{x}{2} & \text{if } x \text{ is even}\\ \frac{x-1}{2}+2^{n-1} & \text{if } x \text{ is odd}\end{cases}$$

Determine the set

$$A=\{x\in\mathbb{N}|(\underbrace{f\circ f\circ\ldots\circ f}_{n})(x)=x\}.$$

Problem 3. Let ω be the incircle of $\triangle ABC$. Denote by I the center of ω. The circle with radius AI and center A intersects circumcircle of $\triangle ABC$ at points P and Q. Prove that line PQ is tangent to ω.

Problem 4. There are 130212 cities in the country of MBL, some pairs of cities are connected by train tracks. Furthermore, you can go between any two cities only by (perhaps multiple) train connections. Prove that the government of MBL may order modernisation of some train tracks so that every city will have an odd number of modern train tracks ending in it.

Problem 5. Let $\mathbb{N}$ denote the set of positive integers. Find all functions $f:\mathbb{N}\to\mathbb{N}$ such that:

(i) The greatest common divisor of the sequence $f(1), f(2), \ldots$ is 1.

(ii) For all sufficiently large integers[5] n, we have $f(n)\neq 1$ and

$$f(a)^n \mid f(a+b)^{a^{n-1}}-f(b)^{a^{n-1}}$$

for all positive integers a and b.

Problem 6. Denote by ω the incircle of $\triangle ABC$. Suppose ω touches the sides $BC,\ AC,\ AB$ at the points $D,\ E,\ F$ respectively. Let I be the center of ω, the segment BI intersects ω at point M. The line tangent to ω at the point M intersects the line AC at the point R. Let the incircle of $\triangle DEF$ touches the segments $EF,\ FD$ at points $D',\ E'$ respectively. The line going through A and perpendicular to ID' intersects line going through B and perpendicular to IE' at the point T. Let J be the A-excenter of $\triangle ABC$. Prove that the points $R,\ T,\ J$ are collinear.

Problem 7. Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying $x_1^2+x_2^2+\ldots+x_n^2=1$. Prove that for every integer $k\geq 2$ there are integers $a_1, a_2, \ldots, a_n$, not all zero, such that $|a_i|\leq k-1$ for all i, and $|a_1x_1+a_2x_2+\ldots+a_nx_n|\leq\frac{(k-1)\sqrt{n}}{k^n-1}$.

[5] that is, there exists positive integer N such that for all positive integers $n>N$ this holds.

Qualifying Quiz - Solutions

Problem 1. Determine all non-zero integers a, for which there exists an integer b such that: $ab|a^2+b^2-a$.

Solution:

Let $a=dx$ and $b=dy$, where $d=\text{nwd}(a,b)$. Because $a \mid a^2+b^2-a$ it follows that $a \mid b^2 \Rightarrow xd \mid d^2y^2 \Rightarrow x \mid dy^2$. But x,y are coprime, so $x \mid d$. On the other hand we write:

$$\frac{a^2+b^2-a}{ab} = \frac{d^2x^2+d^2y^2-dx}{d^2xy}$$

which means that $d \mid x$. Therefore $x=\pm d$, so $a=\pm d^2$. We will consider two cases.

1° $a=d^2$
Then let $b=d$ and see that this satifies the conditions.

2° $a=-d^2$
Our divisibility looks like this:

$$xy \mid x^2+y^2+1 \quad \iff \quad xz=y^2+1 \wedge yw=x^2+1$$

for some $z,w \in \mathbb{Z}_+$ See that x,y must be coprime. Let's take some pair (x,y) that satifies our conditions and assume that it is not $(1,1)$. Suppose that $x>y$ (if $x=y$ then $x=y=1$). Then obviously $w \geqslant x > y \geqslant z$.

We see that $y=\sqrt{xz-1}$ so

$$\sqrt{xz-1}\,w = x^2+1 \qquad xzw^2-w^2=x^4+2x^2+1 \qquad x \mid w^2+1 \wedge w \mid x^2+1$$

. So from pair (x,y) we got pair (w,x). Analogically, we can get a pair (y,z) from (x,y) and therefore we will get (x,y) from (w,x). Hence the connection between pairs satisfying our conditions is symmetric and see that $w+x>x+y>y+z$, so from each pair (apart from $(1,1)$) we can get two pairs: one with greater sum and one with smaller sum. Note that our pairs form a graph whose every connected component is a single chain. See that when taking smaller and smaller pairs we need to stop at some point, as the sum of pair's elements must be positive. But as we have seen, every pair apart from $(1,1)$ can produce a smaller pair in the aforementioned way. So we must stop at $(1,1)$, therefore from every pair we can get $(1,1)$ after some finite number of steps.

Owing to that fact, in our graph there exists only one connected component, as from $(1,1)$ we can get to every other pair. So we have only one chain that is generated by $(1,1)$. Let $a_1=1$, $a_2=1$ and $a_n a_{n-2}=a_{n-1}^2+1$. This is the sequence of elements in our increasing chain of pairs. It can be easily proved by induction (or by Binet's Formula) that:

$$F_{2n+1}F_{2n-3}=F_{2n-1}^2+1$$

where F_n is the n-th Fibonacci number. See that $a_1=F_1$ and $a_2=F_3$. So $a_n=F_{2n-1}$.

Hence in this case, all satisfying a's are $a=-x^2=-F_{2n-1}^2$.

Comments:

Generally speaking, the main idea of all the correct solutions was the same and similar to the one presented above - with the usual difference being the derivation of this chain relation between solutions for negative squares. You, dear reader, are perhaps wondering: "How was I supposed to get that (x,y) is a solution if and only if $\left(\frac{x^2+1}{y},x\right)$ is? Well, one of the alternative ways is to consider the quadratic polynomial in y: $x^2+y^2+1=kxy$ and using Viete formulas get that the second root is also a positive integer with appropriate inequalities needed to solve the problem. This technique is sometimes called *Vieta Jumping*.

Problem 2. Consider a positive integer n and the function $f:\mathbb{N}\to\mathbb{N}$ by

$$f(x)=\begin{cases}\frac{x}{2} & \text{if } x \text{ is even}\\ \frac{x-1}{2}+2^{n-1} & \text{if } x \text{ is odd}\end{cases}$$

Determine the set

$$A=\{x\in\mathbb{N}|(\underbrace{f\circ f\circ\ldots\circ f}_{n})(x)=x\}.$$

Solution 1:

Let's consider x written in a binary expansion. If $2 \mid x$, then x has 0 as the last digit implying that the operation $f(x)$ is equivalent to erasing the last digit. Similarly, if $2 \nmid x$, then x has 1 as the last digit and hence $f(x)$ erases the last digit and adds 2^{n-1} to the obtained number. Therefore, we may conclude that in general $f(x)$ erases the last digit of x and adds it on the n-th place of the binary expansion. This however implies that if $x \in \{0, 1, \ldots, 2^n - 1\}$, then since x can be written using n digits in binary expansion (with leading 0's being allowed), applying function f to it simply rotates its digits. Applying f n times is therefore equivalent to rotating x n times and thus returning it to the initial position. Hence, $0, 1, 2, \ldots, 2^n - 1$ all satisfy $f^{(n)}(x) = x$.

Also, note that due to the properties of the function writing arbitrary positive integer in the form $x + 2^n y$ with x, y being non-negative integers and $x < 2^n$ we obtain that

$$f^{(n)}(x + 2^n y) = f^{(n)}(x) + y = x + y \qquad f^{(n)}(x + 2^n y) = x + 2^n y \quad \iff \quad 2^n y = y \qquad y = 0$$

hence $A = \{1, 2, 3, \ldots, 2^n - 1\}$.

Solution 2:

Note that if x is a non-negative integer less than 2^n, then so is $f(x)$. Indeed, $2 \mid x$ implies $f(x) = \dfrac{x}{2} < x < 2^n$, whereas $2 \nmid x$ results in $f(x) = \dfrac{x-1}{2} + 2^{n-1} < 2^{n-1} + 2^{n-1} = 2^n$. Therefore, $f^{(n)}(x)$ is a non-negative integer less than 2^n as well. Note that,

$$2f(x) = \begin{cases} 2 \cdot \frac{x}{2} = x & \text{if } 2 \mid x \\ 2 \cdot \left(\frac{x-1}{2} + 2^{n-1}\right) = x + 2^n - 1 & \text{if } 2 \nmid x \end{cases}$$

$$2f(x) \equiv 1 \pmod{2^n - 1} \qquad f^{(n)}(x) \equiv 2^n f^{(n)}(x) \equiv 2^{n-1} f^{(n-1)}(x) \equiv \ldots \equiv f(x) \pmod{2^n - 1}$$

Henceforth, due to the fact that $x < 2^n$ implies $f^{(n)}(x) < 2^n$, $0, 1, \ldots, 2^n - 1$ are solutions to the equation $f^{(n)}(x) = x$. Note however that $f^{(n)}(x) = x$ accordingly to all the possibilities of

$$f^{(k)}(x) = \frac{f^{(k-1)}(x)}{2} \qquad \text{or} \qquad f^{(k)}(x) = \frac{f^{(k-1)}(x) - 1}{2} + 2^{n-1}$$

gives rise to 2^n linear equations and hence to at most 2^n solutions. We have already found 2^n solutions and thus there are no more to be found. Therefore, $A = \{1, 2, \ldots, 2^n - 1\}$.

Comments:

There was a multitude of methods applied by applicants during solving this problem. The most popular one was the combination of binary digit rotation and usage of inequalities for $x \geqslant 2^n$ case (usually $x < 2^n \implies f(x) < 2^n$ and $x \geqslant 2^n \implies f(x) < x$, but on the other hand $x \geqslant 2^n \implies f^{(-1)}(x) > x$ alone sufficed) - note that method of linear equations (contrary to the other ones) leads to the solution for the problem if natural numbers are substituted by real numbers.

Except for the approaches presented above, one could prove $x < 2^n$ case by splitting x into sum of powers of two and proving that $f(x)$ is just the sum of values of f on this powers (that's just more algebraic way of binary digit rotation). Interestingly, induction on n was also leading to the solution (details of the proof are left as an exercise to the reader). There were some purely algebraic solutions of this problem, however they required some more effort and being careful not to make any computational errors.

Problem 3. Let ω be the incircle of $\triangle ABC$. Denote by I the center of ω. The circle with radius AI and center A intersects circumcircle of $\triangle ABC$ at points P and Q. Prove that line PQ is tangent to ω.

Solution:

First of all, lets introduce one lemma:

Lemma Let ω and o be the circumcircle and the incircle of $\triangle ABC$ respectively. Let r and R be the radiuses of ω and o respectively. Denote by I the center of ω. Suppose line AI intersects o at point M. Then $AI \cdot MI = 2Rr$.

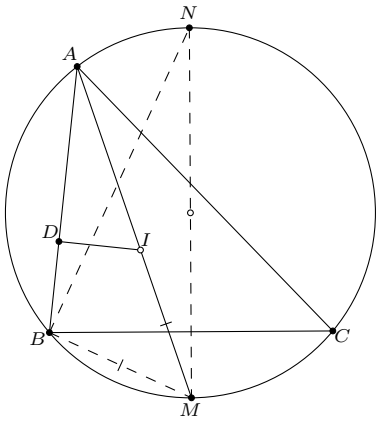
Proof. Let $N \neq M$ be the midpoint of arc BC of o. Let D be the orthogonal projection of I onto the line AB. By Trillium theorem and similarity: $\triangle NBM \sim ADI$ we get:

$$\frac{AI}{ID} = \frac{MN}{MB}$$

so:

$$\frac{AD}{r} = \frac{2R}{IM}$$

and it is equivalent with our thesis. □



Lets come back to the main problem. Let M be the intersection point of AI and o. Since $AP = AQ$, $\sphericalangle PMA = \sphericalangle AMQ$. Hence, using Trillium theorem point I is the incircle of $\triangle PMQ$, denote by r_1 the radius of this circle. Given by **lemma** applied for $\triangle ABC$ and $\triangle PMQ$ we get $2Rr = AI \cdot IM = 2Rr_1$ so $r = r_1$. Therefore, ω is the incircle of $\triangle PMQ$ so PQ is tangent to ω. □

Problem 4. There are 130212 cities in the country of MBL, some pairs of cities are connected by train tracks. Furthermore, you can go between any two cities only by (perhaps multiple) train connections. Prove that the government of MBL may order modernisation of some train tracks so that every city will have an odd number of modern train tracks ending in it.

Solutions:

Note that we may represent our train tracks network by a graph with vertices denoting cities and edges - the train tracks connections. The described property of this network (i.e. you can go between any two cities) implies that the graph is a connected one. However, every connected graph has a spanning tree, hence it suffices to prove that for any tree with even number of vertices we may colour some of its edges such that every vertex is incident with an odd number of coloured edges.
Now that we're past this introduction we may split into 3 different solutions.

Solution 1:

In fact, we may prove slightly more general result.

Lemma:

Suppose that n integers $t_1, t_2, \ldots, t_n$ whose sum is even and a tree on n vertices $T_1, T_2, \ldots, T_n$ are given. Then it is possible to colour some of edges of the given tree such that for every $i \in \{1, 2, \ldots, n\}$ the number of coloured edges incident with T_i has the same parity as t_i.

Proof:

We will prove it by induction on n. The lemma clearly holds for $n = 2$ (we colour the only edge if and only if t_1 is odd). Suppose that the lemma holds for any tree on n verices ($n \geqslant 2$ and that we're given a tree on $n+1$ vertices with t_i's ($i \in \{1, 2, \ldots, n+1\}$) as in the lemma. Every tree has a leaf, i.e. a vertex with only one neighbour. Without loss of generality let's assume that it is a vertex T_{n+1} and that it is connected to T_n.

We colour the edge between T_n and T_{n+1} if and only if t_{n+1} is odd. Since $2 \mid t_1 + t_2 + \ldots + t_{n+1}$, we have $2 \mid t_1 + t_2 + \ldots + t_{n-1} + (t_n - t_{n+1})$. Therefore, ignoring the aforementioned edge and using the inductive assumption on the tree restricted to the first n vertices with $t'_n = t_n - t_{n+1}$ and $t'_i = t_i$ for $i \neq n$ we obtain required colouring of edges proving the lemma.

Now it suffices just to notice that the problem is just a special case of the lemma with $t_1 = t_2 = \ldots = t_n = 1$, hence the problem is solved.

Solution 2:

Since we have an even number of vertices, we can pair them up. For every pair of vertices, there is a path between those two. For any colouring on the given tree if we switch all the coloured edges on the path into

uncoloured and vice versa leaving all the other edges unchanged we don't change the parity of the vertices on the path except for the endpoints. Also, the other vertices are clearly unaffected. Therefore, such an operation changes the parity of coloured incident edges only for the chosen pair. Henceforth, applying repeatedly this operation for all the pairs of points (in the chosen pairing up) yields a desired colouring.

Solution 3:

Let's colour an edge if and only if without this edge the tree splits into two trees with odd number of vertices - it's just simple like that.

Why does it work? Let's choose a vertex of a given tree. Deleting it gives rise to some number of disjoint trees whose sum of vertices is odd. Therefore, there will be an odd number of such trees with odd number of vertices. Using this strategy the chosen vertex will share a coloured edge only with trees with odd number of vertices. Henceforth, it will be incident to an odd number of coloured edges.

Comments:

Looking on these solutions one may think "Aha, so the spanning trees was the magic trick needed to solve this problem." Not necessarily so. In fact the three solutions presented above vary when it comes to their dependence on this fact:

(i) *Solution 1* doesn't need it, however its usage clarifies the proof.

(ii) *Solution 2* is absolutely the same without the assumption of the graph being a tree.

(iii) *Solution 3* doesn't work without the assumption of the graph being a tree.

The most popular approach among the applicants was the path-colour-switching one, more or less in a disguise (for instance you could start from a star and then cut-paste edges as convenient adjusting the colouring using the path-colour-switching strategy). Although noone gave a solution just as it is in the solution 3, many solutions were at some stage of evolution towards it (starting from inductive proof (or equivalently an algorithm) on a rooted tree and ending at the strategies of colouring similar to solution 3 but still using a rooted tree (i.e. colour an edge between a vertex and its son if and only if number of its descendants is odd)). Finally, can you see how solution 3 can be deduced from 2?

Problem 5. Let $\mathbb{N}$ denote the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that:

(i) The greatest common divisor of the sequence $f(1), f(2), \ldots$ is 1.

(ii) For all sufficiently large integers[6] n, we have $f(n) \neq 1$ and

$$f(a)^n \mid f(a+b)^{a^{n-1}} - f(b)^{a^{n-1}}$$

for all positive integers a and b.

Solution:

The following solution will highly rely on the knowledge of Lifting The Exponent Lemma (LTE) and Dirichlet's Theorem, so if you don't know them already, we encourage you to get familiar with these before reading the solution.

Note that $f(1)^n \mid f(b+1) - f(b)$ for all b so one can see that $f(1) \mid \text{nwd}(f(1), f(2), \ldots) = 1$ and hence $f(1) = 1$. Fix $a \in \mathbb{N}$ and let $p \mid f(a)$ be a prime. Since there exists n such that

$$f(a)^n \mid f(a+1)^{a^{n-1}} - f(1)^{a^{n-1}} = f(a+1)^{a^{n-1}} - 1$$

hence $p \nmid f(a+1)$ and so let d be the order of $f(a+1) \mod p$.
By LTE, for $p \neq 2$:

$$nv_p(f(a)) \leqslant v_p(f(a+1)^{a^{n-1}} - 1) = v_p(f(a+1)^d - 1) + v_p\left(\frac{a^{n-1}}{d}\right) \leq$$

$$\leq v_p(f(a+1)^d - 1) + (n-1)v_p(a).$$

[6] that is, there exists positive integer N such that for all positive integers $n > N$ this holds.

But note that for $p = 2$

$$nv_2(f(a)) \leq v_2(f(a+1)^{a^{n-1}} - 1) = v_2(f(a+1) - 1) +$$

$$+ v_2(f(a+1) + 1) + (n-1)v_2(a) - 1.$$

(note that if $2 \mid f(a)$, then $f(a+1)$ is odd, hence we may use LTE). In any case, the result fails to occur if $v_p(f(a)) > v_p(a)$ by taking n sufficiently large. We conclude that $f(a) \mid a$ for all $a \in \mathbb{N}$.
So for large primes p we have $f(p) = 1$ or $f(p) = p$, but only the latter can hold because of the condition that from some point our function does not attain 1 as a value. Observe that

$$p^n = f(p)^n \mid f(b+p)^{p^{n-1}} - f(b)^{p^{n-1}},$$

for fixed $b \in \mathbb{N}$ and p a large prime.

Note that $\text{nwd}(p^{n-1}, p-1) = 1$ hence we may use quite well-known fact that $\phi(x) = x^{p^{n-1}} \mod p$ is a bijection of the set $\{0, 1, 2, \ldots, p-1\}$ (in fact, not only bijection, but also an identity map due to Fermat's Little Theorem). Therefore, the above divisibility implies $p \mid f(b+p) - f(b)$. Repeating the argument k times, shows that $p \mid f(kp+b) - f(b)$ for all positive integers k. By Dirichlet's Theorem, there exists a prime of the form $q = kp + b$. It follows that $p \mid f(b) - b$ and so $f(b) = b$ for all natural numbers b.

Now, it suffices to check that $f(n) = n$ indeed satisfies the required conditions. Obviously, the greatest common divisor of $1, 2, \ldots$ is 1. Also, every divisibility $a^n \mid (a+b)^{a^{n-1}} - b^{a^{n-1}}$ is equivalent to $c^n \mid (c+d)^{c^{n-1}} - d^{c^{n-1}}$ with c, d being coprime positive integers. The latter then follows from LTE.

Comments:

Most of the correct solutions were very similar when it comes to the first two parts of solution (that is $f(1) = 1$ and $f(a) \mid a$). The solutions varied quite significantly at the point of proving $f(a) = a$ and Dirichlet's Theorem wasn't the only advanced theorem used by applicants to prove it – also Bertrand's Postulate, a.k.a. Chebyshev's Theorem and the fact that $\mathbb{Z}/p^k\mathbb{Z}$ is cyclic proved to be useful in the solutions.

Despite the whole heavy machinery used in the solutions, there exists a proof, which omits usage of both Lifting The Exponent Lemma and any of the theorems mentioned above – one could prove the fact that $f(a) \mid a$ using the well-known identity $x^n - y^n = (x-y)(x^{n-1} + \ldots + y^{n-1})$ and proving the $f(a) = a$ follows from considering $q \mid f(a+q) - f(a)$ for q - a prime number with $q > a$. We leave the details of this solution as an exercise for the reader.

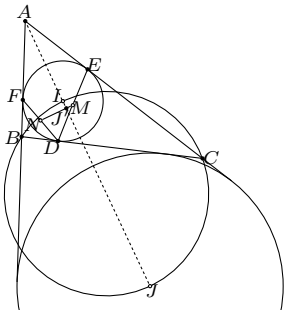
Problem 6. Denote by ω the incircle of $\triangle ABC$. Suppose ω touches the sides $BC,\ AC,\ AB$ at the points $D,\ E,\ F$ respectively. Let I be the center of ω, the segment BI intersects ω at point M. The line tangent to ω at the point M intersects the line AC at the point R. Let the incircle of $\triangle DEF$ touches the segments $EF,\ FD$ at points $D',\ E'$ respectively. The line going through A and perpendicular to ID' intersects line going through B and perpendicular to IE' at the point T. Let J be the A-excenter of $\triangle ABC$. Prove that the points $R,\ T,\ J$ are collinear.

Solution:

First of all we start from some lemmas:

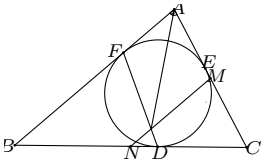
Lemma 1 The incircle of $\triangle ABC$ - ω is tangent to the sides $BC,\ CA,\ AB$ at the points $D,\ E,\ F$ respectively. Denote by J the A-excenter of $\triangle ABC$. Let points $M,\ N$ be the midpoints of segments $DE,\ DF$ respectively. The polar of J in ω is the line MN.

Proof. Denote by I the center of ω. Let Ψ be the inversion with respect ω. We know that $\Psi(B) = N$ and $\Psi(C) = M$. Moreover, the points $B,\ J,\ C,\ I$ lie on the circle so $J' = \Psi(J)$ lies on line MN. On the other hand, the points $A,\ I,\ J$ are collinear so $A,\ I,\ J',\ J$ are collinear. Furthermore, as $MN \parallel EF$ and $AI \perp EF$ it follows $IJ' \perp MN$ so indeed MN is the pole of J in ω. $\square$



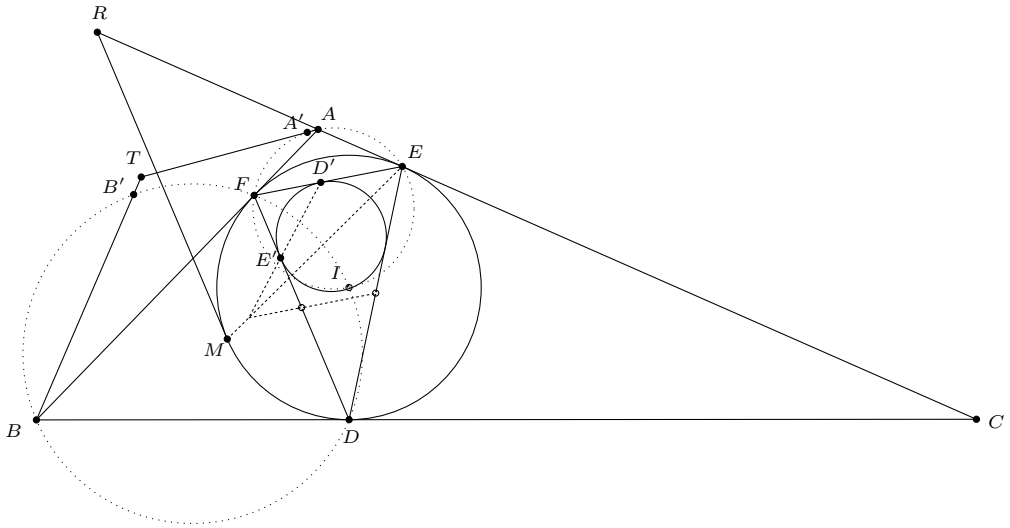
Lemma 2 The incircle of $\triangle ABC$ - ω is tangent to the sides $BC,\ CA,\ AB$ at the points $D,\ E,\ F$ respectively. Denote by $M,\ N$ the midpoints of sides $AC,\ BC$ respectively. Lines $ED,\ MN$ and the angle bisector of $\sphericalangle BAC$ intersect at the one point.

Reader can prove this fact on his/her own.



We can come back to the solution of our main problem. First of all, let $A' = AT \cap ID'$ and $B' = IE' \cap BT$. Since $\sphericalangle AA'I = 90^\circ$ A' lies on circle with diameter AI. This circle is orthogonal to ω so point A' is the image of D' in inversion with respect ω. It follows AA' is the pole of D' in ω. Analogously, BB' is the pole of E' in ω so $D'E'$ is the pole of T in ω.

Furthermore, ME is an angle bisector of $\sphericalangle DFE$ and obviously line ME is the pole of R in ω. It is enough to prove that poles of $R,\ T,\ J$ intersect in one point but it follows from **Lemma 1** and **Lemma 2** for $\triangle DEF$. $\square$



Problem 7. Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying $x_1^2 + x_2^2 + \ldots + x_n^2 = 1$. Prove that for every integer $k \geq 2$ there are integers $a_1, a_2, \ldots, a_n$, not all zero, such that $|a_i| \leq k-1$ for all i, and $|a_1x_1 + a_2x_2 + \ldots + a_nx_n| \leq \frac{(k-1)\sqrt{n}}{k^n-1}$.

Solution:

Without loss of generality we may assume the x_i are nonnegative, by an appropriate (consistent) change in the signs of the a_i.

Now, take the set of $k^n - 1$ nonzero vectors $\mathbf{a}$ formed by each possible combination of $0 \leqslant a_i \leqslant k-1$. Since the x_i are nonnegative, $\mathbf{a.x} \leqslant (k-1)\sum x_i \leqslant (k-1)\sqrt{n\sum x_i^2} = (k-1)\sqrt{n}$

We therefore take each vector in our set and can put its dot product with x into one of the $k^n - 1$ intervals $\left[\frac{\lambda(k-1)\sqrt{n}}{k^n-1}, \frac{(\lambda+1)(k-1)\sqrt{n}}{k^n-1}\right)(\lambda = 0, 1, ..., k^n - 2)$.

If the $\lambda = 0$ interval contains one of our dot products, we are done. Otherwise, by the pigeonhole principle, there exists an interval containing no less than two a vectors. Taking their difference gives a new vector lying in the desired $|\mathbf{a.x}| \leqslant \frac{(k-1)\sqrt{n}}{k^n-1}$ region, so such a vector always exists.

Problems

Olympic Challenge

Younger division

Algebra

Problem 1. Let a, b, c and d be real numbers such that no two of them are equal,

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}=4$$

and $ac = bd$. Find the maximal possible value of

$$\frac{a}{c}+\frac{b}{d}+\frac{c}{a}+\frac{d}{b}.$$

Problem 2. Let $P(x,y), Q(x,y)$ be two-variable polynomials with the coefficients of integer. Suppose that when a_n, b_n are determined for certain integers $a_0,\ b_0$ by $a_{n+1} = P(a_n, b_n),\quad b_{n+1} = Q(a_n, b_n)\ \ (n = 0, 1, 2, \cdots)$ there exists positive integer k such that $(a_1, b_1) \neq (a_0, b_0)$ and $(a_k, b_k) = (a_0, b_0)$.Prove that the number of the lattice points on the segment with end points of (a_n, b_n) and (a_{n+1}, b_{n+1}) is indepedent of n.

Problem 3. Let $0 = x_0 < x_1 < \cdots < x_n = 1$ .Find the largest real number C such that for any positive integer n , we have

$$\sum_{k=1}^{n} x_k^2(x_k - x_{k-1}) > C.$$

Problem 4. Find all functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that $f(f(x) - f(y)) = f(f(x)) - 2x^2 f(y) + f(y^2)$ for all $x, y \in \mathbb{R}$.

Problem 5. A sequence of positive integers $(a_n)_{n\geq 1}$ is of Fibonacci type if it satisfies the recursive relation $a_{n+2} = a_{n+1} + a_n$ for all $n \geq 1$. Is it possible to partition the set of positive integers into an infinite number of Fibonacci type sequences?

Problem 6. Find all polynomials $P(x) \in \mathbb{R}[x]$ such that $P(x - P(x)) = P(x) - P(P(x))$ for all $x \in \mathbb{R}$.

Problem 7. Let $a_1, a_2, \cdots, a_n \geq 1$ and $A = 1+a_1+a_2+\cdots+a_n$. Define $x_0 = 1$, $x_k = \frac{1}{1+a_k x_{k-1}}$. Prove that $x_1+x_2+\cdots+x_n > \frac{n^2 A}{n^2+A^2}$

Problem 8. Let n be a positive even integer, and let $c_1, c_2, \ldots, c_{n-1}$ be real numbers satisfying

$$\sum_{i=1}^{n-1} |c_i - 1| < 1.$$

Prove that polynomial

$$P(x) = 2x^n - c_{n-1}x^{n-1} + c_{n-2}x^{n-2} - \ldots - c_1 x + 2$$

has no real roots.

Problem 9. Let $f(x)$ and $g(x)$ be polynomials with non-negative integer coefficients, and let m be the largest coefficient of f. Suppose that there exist natural numbers $a < b$ such that $f(a) = g(a)$ and $f(b) = g(b)$. Show that if $b > m$, then $f = g$

Combinatorics

Problem 1. There are 2017 distinct points $A_1, A_2, \ldots, A_{2017}$ given on the Cartesian plane. For any permutation $B_1, B_2, \ldots, B_{2017}$ of $A_1, A_2, \ldots, A_{2017}$ define the shadow of a point P as follows: Point P is rotated by 180° around B_1 resulting P_1, point P_1 is rotated by 180° around B_2 resulting P_2, ..., point P_{2016} is rotated by 180° around B_{2017} resulting P_{2017}. Then, P_{2017} is called the shadow of P with respect to the permutation $B_1, B_2, \ldots, B_{2017}$. Let N be the number of different shadows of P up to all permutations of $A_1, A_2, \ldots, A_{2017}$. Determine the maximum value of N.

Problem 2. On a table there are $k \geq 2$ piles having $n_1, n_2, \ldots, n_k$ pencils respectively. A move consists in choosing two piles having a and b pencils respectively, $a \geq b$ and transferring b pencils from the first pile to the second one. Find the necessary and sufficient condition for $n_1, n_2, \ldots, n_k$, such that there exists a succession of moves through which all pencils are transferred to the same pile.

Problem 3. To n people are to be assigned n different houses. Each person ranks the houses in some order (with no ties). After the assignment is made, it is observed that every other assignment assigns at least one person to a house that person ranked lower than in the given assignment. Prove that at least one person received his/her top choice in the given assignment.

Problem 4. It is known that n is a positive integer, $n \leq 144$. Ten questions of type "Is n smaller than a?" are allowed. Answears are given with delay: The answear t the i'th question is give only after $(i+1)$'th question is asked, $i = 1, 2, \ldots, 9$. The answear to the tenth question is given immediately after it is asked. Find a strategy for identifying n.

Problem 5. In a group of several people, some are acquainted with each other and some are not. Every evening, one person invites all of his acquaintances to a party and introduces them to each other. Suppose that after each person has arranged at least one party, some two people are still unacquainted. Prove that they will not be introduced at the next party.

Problem 6. The numbers $1, 2, \ldots, 2006$ are written around the circumference of a circle. A move consists of exchanging two adjacent numbers. After a sequence of such moves, each number ends up 13 positions to the right of its initial position. lf the numbers $1, 2, \ldots, 2006$ are partitioned into 1003 distinct pairs, then show that in at least one of the moves, the two numbers of one of the pairs were exchanged.

Problem 7. Consider $m+1$ horizontal and $n+1$ vertical lines $(m, n \geq 4)$ in the plane forming an $m \times n$ table. Cosider a closed path on the segments of this table such that it does not intersect itself and also it passes through all $(m-1)(n-1)$ interior vertices (each vertex is an intersection point of two lines) and it doesn't pass through any of outer vertices. Suppose A is the number of vertices such that the path passes through them straight forward, B number of the table squares that only their two opposite sides are used in the path, and C number of the table squares that none of their sides is used in the path. Prove that

$$A = B - C + m + n - 1.$$

Problem 8. A fly and k spiders are placed in some vertices of three-dimensional lattice $2017 \times 2017 \times 2017$. One move consists of following: firstly, fly goes to some adjacent vertex or stays where it is and then every spider goes to some adjacent vertex or stays where it is (more than one spider can be in the same vertex). Spiders and fly know where are the others all the time. Find the smallest k so that spiders can catch the fly in finite number of moves, regardless of their initial position.

(Vertices in lattice are adjacent if exactly one coordinate of one vertex is different from the same coordinate of the other vertex, and their difference is equal to 1. Spider catches a fly if they are in the same vertex.)

Problem 9. On a 5×5 board, two players alternately mark numbers on empty cells. The first player always marks 1's, the second 0's. One number is marked per turn, until the board is filled. For each of the nine 3×3 squares the sum of the nine numbers on its cells is computed. Denote by A the maximum of these sums. How large can the first player make A, regardless of the responses of the second player?

Geometry

Problem 1. Denote by O the circumcenter of $\triangle ABC$. The circumcircle of $\triangle AOC$ and $\triangle AOB$ intersects the lines AB and AC at points $A_B,\ A_C$ respectively. Similarly define points $B_A,\ B_C,\ C_A,\ C_B$. Prove that lines $A_BA_C,\ B_AB_C,\ C_AC_B$ have a common point.

Problem 2. Let ABC be an acute scalene triangle with O as its circumcenter. Point P lies inside triangle ABC and satisfies $\sphericalangle PAB = \sphericalangle PBC$ and $\sphericalangle PAC = \sphericalangle PCB$. Point Q lies on line BC and satisfies $QA = QP$. Show that

$$\sphericalangle AQP = 2\sphericalangle OQB.$$

Problem 3. Let ω be the incircle of the $\triangle ABC$ with center I. The line which contains I and is parallel to the line BC intersects the sides AC and AB at the points E and F respectively. Let E' be the symmetric point of E through BI. Let F' be the symmetric point of F through CI. Prove that the line $E'F'$ is tangent to the circle ω.

Problem 4. Two distinct circle ω_1 and ω_2 meet at two distinct points P and Q. A line ℓ intersects ω_1 at the points $A,\ C$ and intersects ω_2 at the points $B,\ D$ such that the points A, B, C and D are all distinct and lie on ℓ in this order. Let T be a point on the line PQ such that P lies between T and Q. Lines $AT,\ BP$ intersect at X and lines $DT,\ CP$ intersect at Y. Finally, let M and N be the midpoints of the segments AD and BC respectively. Prove that $TM,\ PN$ and XY are concurrent.

Problem 5. Let I_b and I_c be the B-excenter and C-excenter of $\triangle ABC$ respectively. Consider a chord PQ of circumcircle of $\triangle ABC$ which is parallel to BC and intersects the segments AB and AC. Suppose the lines BC and AP intersect at R, prove that

$$\sphericalangle I_bQI_c + \sphericalangle I_bRI_c = 180^\circ.$$

Problem 6. On the sides AB and AC of $\triangle ABC$ choose points K and L respectively. Let P and Q be the points laying on the side BC such that $KP \parallel AC$ and $LQ \parallel AB$. The incircle of $\triangle BKP$ touches side AB at the point X. The incircle of $\triangle CLQ$ touches side AC at the point Y. The lines QY and AB intersect at the point M, the lines XP and AC intersect at the point N. Show that if $AX = AY$ then $MN \parallel BC$.

Problem 7. Triangle ABC is given. The line perpendicular to AC and going and through B intersects the circle with diameter AC at the points X and K where X lies closer to B than K. Analogously the line perpendicular to AB and going through C intersects the circle with diameter AB at the points Y and L where Y lies closer to C than L. Prove that the intersection point of XY and KL lies on the line BC.

Problem 8. Let ω be the incircle of $\triangle ABC$ such that $AB = AC$. The circle ω touches sides $BC,\ CA$ and AB at the points $D,\ E$ and F respectively. The line BE intersects ω again at the point X, the line CF intersects ω again at the point Y. Line XY intersects segments $AB,\ AC,\ DF$ and DE at the points $K,\ L,\ P$ and Q respectively. Show that

$$KX = XP = PQ = QY = YL.$$

Problem 9. Let ABC be an acute scalene triangle inscribed in the circle ω. Circle ω, centred at the point O, passes through the points B and C and intersects the sides AB and AC at the points E and D respectively. The point P lies on the arc BC of circumcircle of $\triangle ABC$ with the point A. Prove that the lines BD, CE and OP are concurrent if and only if $\triangle PBD$ and $\triangle PCE$ have the same incenter.

Number Theory

Problem 1. Let $n > 1$ be a positive integer. Tuple $(a_1, a_2, \ldots, a_n)$ of integer numbers greater than 1 is *good* if and only if

$$a_i \Big| \left(\frac{a_1 a_2 \cdots a_n}{a_i} - 1 \right)$$

for $i = 1, 2, \ldots, n$. Prove that there are only finitely many good n-tuples.

Problem 2. Paweł and Ania are playing the following game. First Ania chooses a positive integer x less than 20170. Then Paweł asks her questions about greatest common divisor of numbers $x + n$ and m with n and m being positive integers less than 20170 of his choice. Prove that Paweł can determine this number after 6 such questions.

Problem 3. Let p be a prime number and f an integer polynomial of degree d such that $f(0) = 0, f(1) = 1$ and $f(n)$ is congruent to 0 or 1 modulo p for every integer n. Prove that $d \geqslant p - 1$.

Problem 4. Show that the congruence $x^8 \equiv 16 \pmod{p}$ has a solution for every prime p.

Problem 5. Determine all functions f defined on the natural numbers that take values among the natural numbers for which

$$(f(n))^p \equiv n \mod f(p)$$

for all $n \in \mathbf{N}$ and all prime numbers p.

Problem 6. Let k be a positive integer. Show that there exist infinitely many pairs of integers (a, b) with $|a|, |b| > 1$ such that: $ab + b + a | a^2 + b^2 + k$.

Problem 7. Given $p_1, p_2, \ldots$ be a sequence of integer and $p_1 = 2$. For positive integer n, p_{n+1} is the least prime factor of $np_1^{1!}p_2^{2!}\ldots p_n^{n!} + 1$. Prove that all primes appear in the sequence.

Problem 8. Let p be an odd prime and r an odd natural number.Show that $pr + 1$ does not divide $p^p - 1$.

Problem 9. Let be P be a polynomial with integer coefficients such that for every natural n we have $P(n) > n$. Moreover, if x_k is a sequence that: $x_1 = 1, x_{i+1} = P(x_i)$ then for every natural number N one of x_i is divisible by N. Prove that $P(x) = x + 1$.

Older division

Algebra

Problem 1. Find all pairs (a, b) of a real positive numbers a and b such that

$$\lfloor a\lfloor bn\rfloor\rfloor = n - 1,$$

for all n positive integer.

Problem 2. Find all real polynomials $P(x)$ such that for any three integers $a,\ b$ and c satisfying $a + b + c \neq 0$ number

$$\frac{P(a) + P(b) + P(c)}{a + b + c}$$

is an integer.

Problem 3. Let k, n be integers such that $1 \leqslant k \leqslant n$,and let $a_1,\ a_2,\ \cdots,\ a_k$ be numbers satisfying the following equations.

$$\begin{cases} a_1 + \cdots\cdots + a_k = n \\ a_1^2 + \cdots\cdots + a_k^2 = n \\ \qquad\vdots \\ a_1^k + \cdots\cdots + a_k^k = n \end{cases}$$

Prove that

$$(x + a_1)(x + a_2)\cdots(x + a_k) = x^k + \binom{n}{1}x^{k-1} + \binom{n}{2}x^{k-2} + \cdots + \binom{n}{k}.$$

Problem 4. Let a, b, c be positive real numbers, such that $abc = 1$. Show inequality

$$\frac{1}{a^5(b + 2c)^2} + \frac{1}{b^5(c + 2a)^2} + \frac{1}{c^5(a + 2b)^2} \geq \frac{1}{3}.$$

Problem 5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\left(x^2 + xf(y)\right) = xf(x + y)$$

for all reals x, y.

Problem 6. Find all triples (x, y, z) of real numbers satisfying the system of equations

$$\begin{cases} x^3 = 3x - 12y + 50, \\ y^3 = 12y + 3z - 2, \\ z^3 = 27z + 27x. \end{cases}$$

Problem 7. Find all polynomials $P(x)$ and $Q(x)$ with real coefficients such that

$$P(Q(x)) = P(x)^{2017}$$

for all real numbers x.

Problem 8. Find all polynomials $P(x)$ with real coefficients that satisfy

$$P(x\sqrt{2}) = P(x + \sqrt{1 - x^2})$$

for all real x with $|x| \leq 1$.

Problem 9. For each positive integer n, let $f(n)$ be the maximal natural number such that: $2^{f(n)}$ divides $\displaystyle\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2\cdot i+1}3^i$.

Find all n such that $f(n) = 2017$.

Combinatorics

Problem 1. There is a network of roads in a city each road connecting two crossroads such that one can get from any crossroad to any other. We say that this city is then *connected*. This network has an interesting property such that you can split all the roads into two groups A and B such that one can put directions on roads from A so that city is still connected. Similarly, we can put direction on roads from B (leaving roads from A undirected) so that the city is still connected. Prove that we an put directions on all the roads in the city such that this city is still connected.

Problem 2. Let A be the set of all permutations $a = (a_1, a_2, \ldots, a_{2003})$ of the 2003 first positive integers such that each permutation satisfies the condition: there is no proper subset S of the set $\{1, 2, \ldots, 2003\}$ such that $\{a_k | k \in S\} = S$.

For each $a = (a_1, a_2, \ldots, a_{2003}) \in A$, let $d(a) = \sum_{k=1}^{2003} (a_k - k)^2$.

I. Find the least value of $d(a)$. Denote this least value by d_0. II. Find all permutations $a \in A$ such that $d(a) = d_0$.

Problem 3. *Manhattan distance* between two points (a, b) and (c, d) is defined to be $|a - c| + |b - d|$. Suppose only two distances occur between all pairs of distinct points of some point set. What is the maximal number of points in such a set?

Problem 4. To each vertex of a regular pentagon an integer is assigned, so that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers x, y, z respectively, and $y < 0$, then the following operation is allowed: x, y, z are replaced by $x + y, -y, z + y$ respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps.

Problem 5. Alice and Bob play the following game. To start, Alice arranges the numbers $1, 2, \ldots, n$ in some order in a row and then Bob chooses one of the numbers and places a pebble on it. A player's turn consists of picking up and placing the pebble on an adjacent number under the restriction that the pebble can be placed on the number k at most k times. The two players alternate taking turns beginning with Alice. The first player who cannot make a move loses. For each positive integer n, determine who has a winning strategy.

Problem 6. Let $T = \{1, 2, 3, \ldots, 14, 15\}$. Say that a subset S of T is handy if the sum of all the elements of S is a multiple of 5. For example, the empty set is handy (because its sum is 0) and T itself is handy (because its sum is 120). Compute the number of handy subsets of T.

Problem 7. Paweł has put the participants of MBL in three lecture rooms. He is only allowed to transfer people from one room to another so that the number of participants in the latter is doubled.

(a) Prove that Paweł can always empty one of the rooms.

(b) Can he put all the participants in one room?

Problem 8. A country has 2016 cities. Every pair of cities is connected by a road. Each road is owned by one of 10 companies. Prove that there must be a way to travel in a circuit of odd length along a sequence of roads that are all owned by a single company.

Problem 9. Suppose that each of the vertices of a simple graph is equipped with an indicator light and bubble. Each vertex's button simultaneously toggles the states of all its neighbours as well as its own state. Initially all lights are off. Prove that it is possible to turn on all the lights.

Geometry

Problem 1. In acute $\triangle ABC$, let o be its circumcircle and ω be its incircle. Denote by O and I center of o and ω respectively. Tangents at the points B and C to o meet at the point L. The circle ω touches the side BC at the point D. The line AY is perpendicular to BC at the point Y and AO meets BC at the point X. Suppose the line OI meets o at the points P and Q. Prove that if $A,\ D$ and L are collinear, then the points $P,\ Q,\ X$ and Y are concyclic.

Problem 2. Let O and I be the circumcenter and incenter of triangle ABC respectively. The perpendicular from I to OI meets BC and the external bisector of angle A at points X and Y respectively. In what ratio does I divide the segment XY?

Problem 3. Let τ be the circumcircle of $\triangle ABC$. Let the points $D,\ E$ lie on the sides $AC,\ AB$ respectively. Let $K = BD \cap \tau$, $L = CE \cap \tau$. Tangents to τ at the points $B,\ C$ intersect the line DE at the points $P,\ Q$ respectively. Prove that PL and QK intersect on τ.

Problem 4. Given a $\triangle ABC$ with $\sphericalangle BAC = 90^\circ$ and two perpendicular lines x and y passing through the vertex A. For an arbitrary point X lying on the line x define y_b and y_c as the reflections of the line y about the line XB and XC respectively. Let Y be the common point of the lines y_b and y_c. Find the locus of the points Y.

Problem 5. Let $ABCD$ be the quadrilateral inscribed in the circle ω. Line l is tangent to the circle ω at the point T. The lines $AB,\ BC,\ CD$ and AD intersect the line l at the points $Q,\ X,\ P$ and Y respectively. Show that if $TY = TX$ then $TP = TQ$.

Problem 6. Let H_B and H_C be the foots of the altitudes from the vertexes B and C in $\triangle ABC$ respectively. Denote by Γ circumcircle of $\triangle ABC$. Prove that circumcircle of triangle bounded by the lines tangent to Γ at the points B and C and the line H_BH_C is tangent to Γ.

Problem 7. A scalene $\triangle ABC$ and its incircle ω are given. Using only a ruler and drawing at most eight lines, rays or segments, construct points $A',\ B'$ and C' on ω such that the rays $B'C',\ C'A'$ and $A'B'$ pass through the points $A,\ B$ and C respectively.

Problem 8. Let ω be the incircle of $\triangle ABC$. The circle ω touches sides BC and AB at the points D and F respectively. Denote by D' the reflection of the point D with respect to the point C. The circle passing through the points B and C is tangent to ω at the point T. Let J be the A-excenter of $\triangle ABC$. Prove that the points $T,\ F,\ B,\ J$ and D' lie on the one circle.

Problem 9. Let P be any point inside $\triangle ABC$. Denote by H_C and H_B the orthocenters of $\triangle PAB$ and $\triangle PAC$ respectively. The line PA cuts circumcircle of $\triangle BPC$ again at the point Q. The point H is the projection of Q onto the line BC. Prove that $AH \perp H_BH_C$.

Number Theory

Problem 1. Find all $f(x) \in \mathbb{Z}[x]$ such that $f(p)|(2^p - 2)$ for all prime number p.

Problem 2. Let P be a polynomial with integer coefficients such that $P(0) = 0$ and

$$\text{nwd}(P(0), P(1), P(2), \ldots) = 1.$$

Show there are infinitely many n such that

$$\text{nwd}(P(n) - P(0), P(n+1) - P(1), P(n+2) - P(2), \ldots) = n.$$

Problem 3. Show that there exist only finitely many positive integers n such that

$$\varphi(n) < n^{\frac{1}{2017}}.$$

Problem 4. Let $n > 1$ be a given integer. Prove that infinitely many terms of the sequence $(a_k)_{k \geq 1}$, defined by

$$a_k = \left\lfloor \frac{n^k}{k} \right\rfloor,$$

are odd. (For a real number x, $\lfloor x \rfloor$ denotes the largest integer not exceeding x.)

Problem 5. Define the sequence $a_1, a_2, a_3, \ldots$ by $a_1 = 1$ and, for $n > 1$,

$$a_n = a_{\lfloor n/2 \rfloor} + a_{\lfloor n/3 \rfloor} + \ldots + a_{\lfloor n/n \rfloor} + 1.$$

Prove that there are infinitely many n such that $a_n \equiv n \pmod{2^{2017}}$.

Problem 6. Let the function $f : N^* \to N^*$ such that (1) $(f(m), f(n)) \leq (m, n)^{2014}, \forall m, n \in N^*$; (2) $n \leq f(n) \leq n+2014, \forall n \in N^*$ Show that: there exists the positive integers N such that $f(n) = n$, for each integer $n \geq N$.

Problem 7. For a prime p, a subset S of residues modulo p is called a sum-free multiplicative subgroup of $\mathbb{F}_p$ if • there is a nonzero residue α modulo p such that $S = \{1, \alpha^1, \alpha^2, \ldots\}$ (all considered mod p), and • there are no $a, b, c \in S$ (not necessarily distinct) such that $a + b \equiv c \pmod{p}$. Prove that for every integer N, there is a prime p and a sum-free multiplicative subgroup S of $\mathbb{F}_p$ such that $|S| \geq N$.

Problem 8. Let $n \geq 2$ be an integer. Prove that if $k^2 + k + n$ is prime for all integers k such that $0 \leq k \leq \sqrt{\frac{n}{3}}$, then $k^2 + k + n$ is prime for all integers k such that $0 \leq k \leq n - 2$.

Problem 9. Let $a > 1$ be a positive integer and $f \in \mathbb{Z}[x]$ with positive leading coefficient. Let S be the set of integers n such that

$$n \mid a^{f(n)} - 1.$$

Prove that S has density 0; that is, prove that $\lim_{n \to \infty} \frac{|S \cap \{1, \ldots, n\}|}{n} = 0$.

Mathematical Matches

Younger Match

Algebra

Problem 1. Find all triples (f, g, h) of injective functions from the set of real numbers to itself satisfying

$$f(x + f(y)) = g(x) + h(y)$$
$$g(x + g(y)) = h(x) + f(y)$$
$$h(x + h(y)) = f(x) + g(y)$$

for all real numbers x and y. (We say a function F is injective if $F(a) \neq F(b)$ for any distinct real numbers a and b.)

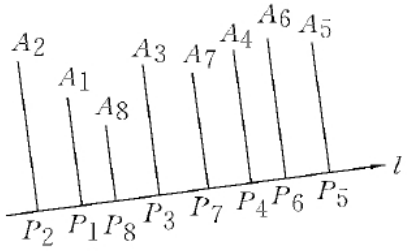
Problem 2. Find all surjective functions $f : (0, +\infty) \to (0; +\infty)$ satisfying

$$2xf(f(x)) = f(x)(x + f(f(x))).$$

Combinatorics

Problem 1. Let S be a set of 2017 points in the coordinate plane, no two of which have the same $x-$ or $y-$coordinate. For any two points $P, Q \in S$, consider the rectangle with one diagonal PQ and the sides parallel to the axes. Denote by W_{PQ} the number of points of S lying in the interior of this rectangle. Determine the maximum N such that, no matter how the points of S are distributed, there always exist points P, Q in S with $W_{PQ} \geq N$.

Problem 2. Assume that $A_1, A_2, \ldots, A_8$ are eight points taken arbitrarily on a plane. For a directed line l taken arbitrarily on the plane, assume that projections of $A_1, A_2, \ldots, A_8$ on the line are $P_1, P_2, \ldots, P_8$ respectively. If the eight projections are pairwise disjoint, they can be arranged as $P_{i_1}, P_{i_2}, \ldots, P_{i_8}$ according to the direction of line l. Thus we get one permutation for $1, 2, \ldots, 8$, namely, $i_1, i_2, \ldots, i_8$. In the figure, this permutation is $2, 1, 8, 3, 7, 4, 6, 5$. Assume that after these eight points are projected to every directed line on the plane, we get the number of different permutations as $N_8 = N(A_1, A_2, \ldots, A_8)$. Find the maximal value of N_8.



Problem 3. Given a finite sequence of real numbers $a_1, a_2, \ldots, a_n$ $(*)$, we call a segment $a_k, \ldots, a_{k+l-1}$ of the sequence $(*)$ a *dragon* and a_k is a *head* of the dragon if the arithmetic mean of $a_k, \ldots, a_{k+l-1}$ is greater than 2017 (especially if a single item $a_m > 2017$, we still regard a_m as a dragon). Suppose that there is at least one dragon among the sequence $(*)$, show that the arithmetic mean of all those items of sequence $(*)$ that could be head of a certain dragon (that is, the arithmetic mean of all the heads, counted with multiplicities) is greater than 2017.

Geometry

Problem 1. Let $ABCD$ be a convex quadrilateral satisfying $AC = BD = AB$. Let M and N be the midpoints of the segments AD and BC respectively. Denote by T the common point of the diagonals. Prove that the line MN passes through the touching points of the incircle of $\triangle ATB$ with the sides AT and TB.

Problem 2. Equilateral triangle ABC is given. Let o be the circumcircle and ω be the incircle of $\triangle ABC$ with common center O. Let P and Q be the points lying on the sides AC and AB respectively, such that O lies on PQ. Let γ_b and γ_c be the circles with diameters BP and CQ respectively. Show that one of the intersection points of γ_b and γ_c lies on o and the other lies on ω.

Problem 3. Let H be the ortocenter of $\triangle ABC$. The point P lies on circumcircle of $\triangle ABC$ and the point M is the midpoint of the side BC. Let T be the projection of H onto line AP. Prove that

$$MT = MP.$$

Number Theory

Problem 1. Let N be a natural number. Does there exist an increasing arithmetic progression which has N terms and the sum of every term's digits is constant?

Problem 2. Prove that if $n \cdot \phi(n) = m \cdot \phi(m)$ then $m = n$

Problem 3. For an integer $x \geqslant 1$, let $p(x)$ be the least prime that does not divide x, and define $q(x)$ to be the product of all primes less than $p(x)$. In particular, $p(1) = 2$. For x having $p(x) = 2$, define $q(x) = 1$. Consider the sequence $x_0, x_1, x_2, \ldots$ defined by $x_0 = 1$ and

$$x_{n+1} = \frac{x_n p(x_n)}{q(x_n)}$$

for $n \geqslant 0$. Find all n such that $x_n = 1995$.

Middle Match

Algebra

Problem 1. Show that if the integers a_1, a_2,..., a_m are nonzero and for each $k = 0,\ 1, \ldots, n$ $(n < m-1)$, $a_1 + a_2 2^k + a_3 3^k + \ldots + a_m m^k = 0$, then the sequence $a_1, \ldots, a_m$ contains at least $n+1$ pairs of consecutive terms having opposite signs.

Problem 2. Find all $a \in \mathbb{R}$ for which there exists a non-constant function $f : (0, 1] \to \mathbb{R}$ such that

$$a + f(x + y - xy) + f(x)f(y) \leqslant f(x) + f(y)$$

for all $x, y \in (0, 1]$.

Combinatorics

Problem 1. In an $m \times n$ rectangular grid, where m and n are odd integers, 1×2 dominoes are initially placed so as to exactly cover all but one of the 1×1 squares at one corner of the grid. It is permitted to slide a domino towards the empty square, thus exposing another square. Show that by a sequence of such moves, we can move the empty square to any corner of the rectangle.

Problem 2. Paweł and Ania eat a pizza of $2n$ pieces. Each piece contains a distinct amount of olives between 1 and $2n$. Paweł eats the first piece, and the two players alternately eat a piece neighbouring with an eaten piece. However, neither Paweł nor Ania like olives, so they will choose pieces that minimizes the total amount of olives they eat. For each arrangement σ of the olives, let $s(\sigma)$ the minimal amount of olives that Paweł can eat, considering that both play in the best way possible. Let $S(n)$ the maximum of $s(\sigma)$, considering all arrangements.

a) Prove that $n^2 - 1 + \lfloor \frac{n}{2} \rfloor \leq S(n) \leq n^2 + \lfloor \frac{n}{2} \rfloor$

b) Prove that $S(n) = n^2 - 1 + \frac{n}{2}$ for each even n.

Problem 3. A piece at the origin of the coordinate plane is given. Gábor and Mateusz act as following. First, Gábor marks on a lattice point, on which the piece cannot be anymore put. Then Mateusz moves the piece from the point $(x,\ y)$ to the point $(x+1,\ y)$ or $(x,\ y+1)$, at most k times. He cannot move the piece to a marked point. If Gábor wins when Mateusz can't move any pieces, then find all possible integers k such that Gábor will win in a finite number of moves, regardless of how Mateusz moves the piece.

Geometry

Problem 1. Given is an acute $\triangle ABC$. Choose T - an arbitrary point at the side AB. Let N be the midpoint of the segment AC. The foots of the perpendiculars from the point A to the segments TC and TN are the points P and Q respectively. Prove that center of the circumcircle of $\triangle PQN$ lies on a fixed line for all the points T from the side AB.

Problem 2. Let $\tau_1,\ \tau_2$ be the intersecting circles with centers $O_1,\ O_2$ respectively. Point A lies on τ_1, point B lies on τ_2 and AB is an ex-tangent line of τ_1 and τ_2. Suppose C is one of the intersection points of τ_1 and τ_2 nearer to AB. Let D be the point laying on O_1O_2 such that the line BD is perpendicular to AC. Prove that $\sphericalangle BCD = 90^\circ$.

Problem 3. Let ABC be a triangle with incenter I. Let D be a point on side BC and let ω_B and ω_C be the incircles of $\triangle ABD$ and $\triangle ACD$ respectively. Suppose that ω_B and ω_C are tangent to segment BC at points E and F respectively. Let P be the intersection of segment AD with the line joining the centers of ω_B and ω_C. Let X be the intersection point of the lines BI and CP and let Y be the intersection point of the lines CI and BP. Prove that lines EX and FY meet on the incircle of $\triangle ABC$.

Number Theory

Problem 1. Show that there exists a bijective function $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that for all $m, n \in \mathbb{N}_0$:

$$f(3mn + m + n) = 4f(m)f(n) + f(m) + f(n).$$

Problem 2. Let A be a finite set of positive integers. Prove that there exists a finite set B of positive integers such that $A \subseteq B$ and

$$\prod_{x \in B} x = \sum_{x \in B} x^2.$$

Problem 3. Let m be a positive integer. Find all bounded sequences of integers $a_1, a_2, a_3, ...$ for which $a_n + a_{n+1} + a_{n+m} = 0$ for all $n \in \mathbb{N}$.

Older Match

Algebra

Problem 1. Let $x,\ y$ and z be a positive real numbers. Show that

$$\frac{xy}{z}+\frac{yz}{x}+\frac{zx}{y}>2\sqrt[3]{x^3+y^3+z^3}.$$

Problem 2. Let $a_1, a_2...a_n$ be non-negative reals, not all zero. Show that that

a) The polynomial $p(x)=x^n-a_1x^{n-1}+...-a_{n-1}x-a_n$ has preceisely 1 positive real root R.

b) let $A=\sum_{i=1}^{n}a_i$ and $B=\sum_{i=1}^{n}ia_i$. Show that $A^A\leqslant R^B$.

Geometry

Problem 1. Subsets of the set $S=\{1,2,...,n\}$ are coloured with 2017 colors. We call a pair of disjoint subsets A, B of S *good* if they have the same color as $A\cup B$. Prove that if n is large enough, then we can find a good pair of subsets in S.

Problem 2. For a point P on the plane, denote by $\|P\|$ the distance to its nearest lattice point. Prove that there exists a real number $L>0$ satisfying the following condition:

For every $\ell>L$, there exists an equilateral triangle ABC with side-length ℓ and $\|A\|,\|B\|,\|C\|<10^{-2017}$.

Problem 3. A figure Φ composed of unit squares has the following property: if the squares of an $m\times n$ rectangle (m,n are fixed) are filled with numbers whose sum is positive, the figure Φ can be placed within the rectangle (possibly after being rotated) so that the sum of the covered numbers is also positive. Prove that a number of such figures can be put on the $m\times n$ rectangle so that each square is covered by the same number of figures.

Combinatorics

Problem 1. Let H be the orthocenter of $\triangle ABC$. The lines l_1 and l_2 pass through H and $l_1\perp l_2$. The line l_1 intersects the lines AB and BC at the points K and P respectively. The line l_2 intersects the lines AC and BC at the points L and Q respectively. The line passing through the point Q and parallel to the line AB intersects the line passing through P and parallel to the line AC at the point T. Prove that the points $K,\ T$ and L are collinear.

Problem 2. Let ABC be an acute triangle with the orthocenter H. On the segments $HA,\ HB$ and HC, we choose three points $A_1,\ B_1$ and C_1 respectively such that

$$\sphericalangle BA_1C=\sphericalangle CB_1A=\sphericalangle AC_1B=90^\circ.$$

Denote by $A_2:=BC_1\cap CB_1,\ B_2:=CA_1\cap AC_1$ and $C_2:=AB_1\cap BA_1$.

a) Prove that the sides of the hexagon $A_1B_2C_1A_2B_1C_2$ are tangent to the one circle.

b) Let J be a incenter of the hexagon $A_1B_2C_1A_2B_1C_2$. Prove that the lines A_1A_2, B_1B_2, C_1C_2 and HJ are concurrent.

Problem 3. Given a $\triangle ABC$ has incenter at the point I. A line passes through I and intersects circumcircles of $\triangle IBC,\ \triangle ICA$ and $\triangle IAB$ again at the points $D,\ E$ and F respectively. Prove that the Euler lines of $\triangle DBC,\ \triangle ECA$ and $\triangle FAB$ are either concurrent or parallel.

Number Theory

Problem 1. Let $\varphi(n)$ denote the number of positive integers less than n that are relatively prime to n. Prove that there exists a positive integer m for which the equation $\varphi(n)=m$ has at least 2015 solutions in n.

Problem 2. Let p be the largest prime factor of $A=2^{2^n}+1$ $(n>3)$ Prove that $p\geq(n+2)2^{n+4}+1$

Problem 3. Let p be an odd prime and n a positive integer. In the coordinate plane, seven distinct points with integer coordinates lie on a circle with diameter of length p^n. Prove that there exists a triangle with vertices at three of the given points such that the squares of its side lengths are integers divisible by p^{n+1}.

Relays

Problem 1. Let N be the greatest integer multiple of 8, no two of whose digits are the same. What is the remainder when N is divided by 1000?

Problem 2. How many positive integers have exactly three proper divisors, each of which is less than 50?

Problem 3. Let S_i be the sum of the first i terms of the arithmetic sequence $3, 7, 11, 15, \ldots$. What are possible values of the expression

$$\frac{S_i}{i}(j-k)+\frac{S_j}{j}(k-i)+\frac{S_k}{k}(i-j)$$

subject to all possibilities of i, j, k

Problem 4. Let $M(n) = \{-1, -2, \ldots, -n\}$. For every non-empty subset of $M(2017)$ we consider the product of its elements. How big is the sum over all these products?

Problem 5. We call an isosceles trapezoid $PQRS$ interesting, if it is inscribed in the unit square $ABCD$ in such a way, that on every side of the square lies exactly one vertex of the trapezoid and that the lines connecting the midpoints of two adjacent sides of the trapezoid are parallel to the sides of the square.

Find the areas of all interesting trapzoids.

Problem 6. Jane is 25 years old. Dick is older than Jane. In n years, where n is a positive integer, Dick's age and Jane's age will both be two-digit number and will have the property that Jane's age is obtained by interchanging the digits of Dick's age. Let d be Dick's present age. How many ordered pairs of positive integers (d, n) are possible?

Problem 7. Let $A_1, A_2, A_3, \ldots, A_{12}$ be the vertices of a regular dodecagon. How many distinct squares in the plane of the dodecagon have at least two vertices in the set $\{A_1, A_2, A_3, \ldots, A_{12}\}$?

Problem 8. The sequence a_n is defined by $a_1 = 1, a_2 = 2$ and every element is a sum of all the previous ones. What is the value of a_{12}?

Problem 9. Let $\star$ be an operation defined in the set of nonnegative integers with the following properties: for any nonnegative integers x and y,

(a) $(x+1)\star 0 = (0\star x)+1$

(b) $0\star(y+1) = (y\star 0)+1$

(c) $(x+1)\star(y+1) = (x\star y)+1$.

If $123\star 456 = 789$, find $246\star 135$

Problem 10. Now run and find (Grzegorz?) and play with him the following game. At the beginning he will tell you a positive integer n and you choose whether you want to go first. Numbers $1, 2, \ldots, 10$ will be written on the paper. Alternately, you put $+$ or $-$ before the numbers. The first player wins if and only if the absolute value of the obtained sum doesn't exceed n.

Problem 11. How many 6-tuples (a, b, c, d, e, f) of natural numbers are there for which $a > b > c > d > e > f$ and $a + f = b + e = c + d = 22$ are simultaneously true?

Problem 12. In decimal representation

34!=295232799039a041408476186096435b0000000.

Find the numbers a and b.

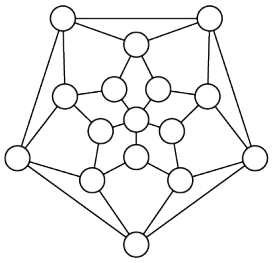
Problem 13. Let x_1, x_2, …, x_{10} be 10 numbers. Suppose that $x_i + 2x_{i+1} = 1$ for each i from 1 through 9. What is the value of $x_1 + 512x_{10}$?

Problem 14. We are given a biased coin with heads ending up with probability $\frac{2}{7}$ and tails otherwise. We toss it until we get heads. What is the expected number of tosses.

Problem 15. In the interior of the square $ABCD$, the point P lies in such a way that $\sphericalangle DCP = \sphericalangle CAP = 25°$. Find the value of $\sphericalangle PBA$.

Problem 16. A computer generates even integers half of the time and another computer generates even integers a third of the time. If a_i and b_i are the integers generated by the computers, respectively, at time i, what is the probability that $a_1b_1 + a_2b_2 + \cdots + a_6b_6$ is an even integer?

Problem 17. At each of the sixteen circles in the network below stands a student. A total of 3360 coins are distributed among the sixteen students. All at once, all students give away all their coins by passing an equal number of coins to each of their neighbors in the network. After the trade, all students have the same number of coins as they started with. Find the number of coins the student standing at the center circle had originally.



Problem 18. Let $f(x) = \dfrac{9^x}{9^x + 3}$. Compute $\displaystyle\sum_k f\left(\frac{k}{2017}\right)$, where k goes over all integers k between 1 and 2017 which are coprime to 2017.

Problem 19. A function $f : \mathbb{Z} \to \mathbb{R}$ satisfies the equation:

$$f(n) = \begin{cases} n - 100 & \text{for } n > 2000 \\ f(f(n + 101)) & \text{for } n \leq 2000 \end{cases}$$

Find $f(1)$.

Problem 20. Go play game with Grzegorz. You will be given two piles of (cookies or something like that) In each move you may eat one of the sweets from one pile, one from each of the piles or move one sweet from one pile to the another. Whoever empties the piles, wins.

Problem 21. Let S_1, S_2, ..., S_{125} be 125 sets of 5 numbers each, comprising 625 distinct numbers. Let m_i be the median of S_i. Let M be the median of m_1, m_2, ..., m_{125}. What is the greatest possible number of the 625 numbers that are less than M?

Problem 22. Consider the following 50-term sums:
$S = \frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + ... + \frac{1}{99\cdot 100}$,
$T = \frac{1}{51\cdot 100} + \frac{1}{52\cdot 99} + ... + \frac{1}{99\cdot 52} + \frac{1}{100\cdot 51}$.
Express $\frac{S}{T}$ as an irreducible fraction.

Problem 23. Define a sequence $u_n, n = 0, 1, 2, 3...$ as follows: $u_0 = 0$, $u_1 = 1$ and, for each $n \geqslant 1, u_{n+1}$ is the smallest positive integer such that $u_{n+1} > u_n$ and $u_1, u_2, ..., u_n, u_{n+1}$ contains no three elements that are in arithmetic progression. Find u_{100}.

Problem 24. Determine all positive integers n such that the number $n(n + 2)(n + 4)$ has at most 15 positive divisors.

Problem 25. The bisectors of the angles A and B in triangle ABC meet the sides BC and CA at the points D and E, respectively. Assuming that the angle A equals 45° and $AE + BD = AB$, determine the angle B.

Problem 26. Consider a strip of 12 fieds, numbered from left to right with the integers 1 to n in ascending order. Each of the fields is colored with one of the colors 1, 2 or 3. Even-numbered fields can be colored with any color. Odd-numbered fields are only allowed to be colored with the odd colors 1 and 3. How many such colorings are there such that any two neighboring fields have different colors?

Problem 27. How many ways can the numbers $1, 2, 3, 4, 5$ be placed in a circular table with 6 seats, such that no two consecutive numbers are placed together? Any two arrangements obtained by rotation or reflection are considered to be the same.

Problem 28. Find all prime numbers p such that $2^p + p^2$ is also a prime number.

Problem 29. Let $f(n)$ denote an integer which is the closest to number $\sqrt{n}$. Compute

$$\frac{1}{f(1)} + \frac{1}{f(2)} + \ldots + \frac{1}{f(10000)}$$

Problem 30. Now run and find Grzegorz and play with him the following game. You are given number $N = 2310$. You alternately write composite divisors of N, but every time you choose one you must make sure that all the written divisors are not coprime and also there are no two divisors such that one divides the other. You decide who goes first. But remember you have only one chance.

Problem 31. Twenty-seven balls labelled from 1 to 27 are distributed in three bowls: red, blue, and yellow. What is the average of the possible values of the number of balls in the red bowl if the average labels in the red, blue and yellow bowl are 15, 3, and 18, respectively?

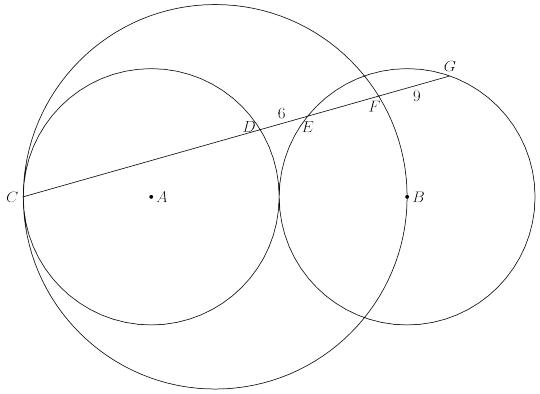
Problem 32. If $d(n)$ denotes the number of positive divisors of n, then evaluate

$$d(1) + d(2) + \ldots + d(24)$$

Problem 33. We define the weight W of a positive integer as follows: $W(1) = 0$, $W(2) = 1$, $W(p) = 1 + W(p + 1)$ for every odd prime p, $W(c) = 1 + W(d)$ for every composite c, where d is the greatest proper factor of c. Compute the greatest possible weight of a positive integer less than 100.

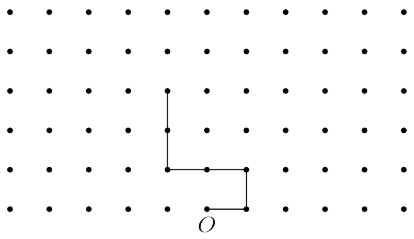
Problem 34. We say that some positive integer m *covers* the number 217 if $2, 1, 7$ appear in this order as digits of m (For instance, 217 is covered by **21**33267**9**8, but not by 215993698). Let $k(n)$ be the number of positive integers that cover 217 and have exactly n digits, all different from 0. What is the remainder of $k(40)$ in division by 72?

Problem 35. In the diagram below, the circle with center A is congruent to and tangent to the circle with center B. A third circle is tangent to the circle with center A at point C and passes through point B. Points C, A, and B are collinear. The line segment $\overline{CDEFG}$ intersects the circles at the indicated points. Suppose that $DE = 6$ and $FG = 9$. Find AG.



Problem 36. Alice, Beth, Carla, Dana, and Eden play a game in which each of them simultaneously closes her eyes and randomly chooses two of the others to point at (one with each hand). A participant loses if she points at someone who points back at her; otherwise, she wins. Find the probability that all five girls win.

Problem 37. The following figure shows a walk of length 6:



This walk has three interesting properties:

It starts at the origin, labelled O. Each step is 1 unit north, east, or west. There are no south steps. The walk never comes back to a point it has been to.

Let's call a walk with these three properties a northern walk. There are 3 northern walks of length 1 and 7 northern walks of length 2. How many northern walks of length 6 are there?

Problem 38. An integer $n \geqslant 3$ is called special if it does not divide $(n-1)!\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\right)$. Find all special numbers n such that $10 \leqslant n \leqslant 50$.

Problem 39. The sequence a_0, a_1, a_2, ... satisfies the recurrence equation

$$a_n = 2a_{n-1} - 2a_{n-2} + a_{n-3}$$

for every integer $n \geq 3$. If $a_{20} = 1$, $a_{25} = 10$, and $a_{30} = 100$, what is the value of a_{1331}?

Problem 40. You will be given a circle divided by 11 points into equal arcs. You and Grzegorz (you choose who goes first) alternately choose points to remove. The one after whose move all triangles are obtuse wins.

Solutions

Olympic Challenge

Algebra

Younger division

Problem 1. Let a, b, c and d be real numbers such that no two of them are equal,

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}=4$$

and $ac = bd$. Find the maximal possible value of

$$\frac{a}{c}+\frac{b}{d}+\frac{c}{a}+\frac{d}{b}.$$

Solution:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}=4 \qquad (a+c)(b+d)=4ac$$

Let's note that since a, b, c, d are pairwise distinct numbers, hence $ac = bd < 0$, since otherwise $(a + c)(b + d) = 4ac > 0$

$$(a+c)(b+d)=|a+c||b+d|>2\sqrt{ac}\cdot 2\sqrt{bd}=4ac$$

Due to this fact as well as AM-GM inequality we obtain

$$\frac{a^2+b^2+c^2+d^2}{ac}=\frac{(a+c)^2+(b+d)^2}{ac}-4\leq -2\frac{(a+c)(b+d)}{ac}-4=-12$$

The equality here appears for instance for $(a, b, c, d) = (1, -1, -3 - 2\sqrt{2}, 3 + 2\sqrt{2})$.

Problem 2. Let $P(x, y), Q(x, y)$ be two-variable polynomials with the coefficients of integer.Suppose that when a_n, b_n are determined for certain integers $a_0,\ b_0$ by $a_{n+1} = P(a_n, b_n),\quad b_{n+1} = Q(a_n, b_n)\ \ (n = 0, 1, 2, \cdots)$ there exists positive integer k such that $(a_1, b_1) \neq (a_0, b_0)$ and $(a_k, b_k) = (a_0, b_0)$.Prove that the number of the lattice points on the segment with end points of (a_n, b_n) and (a_{n+1}, b_{n+1}) is indepedent of n.
Solution:

Let $d_l = (a_{l+1} - a_l, b_{l+1} - b_l)$. Since for every polynomial with integer coefficients $W(x)$ and integers x, y, we have $x - y \mid W(x) - W(y)$, hence considering $P(x, b_k)$ as a polynomial in x we obtain

$$a_{l+1}-a_l \mid P(a_{l+1},b_k)-P(a_l,b_l)$$

Also, considering $P(a_{l+1}, x)$ as a polynomial in x, we obtain

$$b_{l+1}-b_l \mid P(a_{l+1},b_{l+1})-P(a_{l+1},b_l)$$

Therefore,

$$d_l \mid P(a_{l+1},b_{l+1})-P(a_{l+1},b_l)+P(a_{l+1},b_l)-P(a_l,b_l)=P(a_{l+1},b_{l+1})-P(a_l,b_l)=a_{l+2}-a_{l+1}$$

Analogously, $d_l \mid b_{l+2} - b_{l+1}$, hence $d_l \mid d_{l+1}$. Thus,

$$d_0 \mid d_1 \mid \ldots \mid d_k = d_0 \qquad d_0 = d_1 = \ldots = d_{k-1}$$

This however implies that the number of lattice points on the segment with endpoints at (a_n, b_n) and (a_{n+1}, b_{n+1}) is independent of n.

Problem 3. Let $0 = x_0 < x_1 < \cdots < x_n = 1$. Find the largest real number C such that for any positive integer n , we have

$$\sum_{k=1}^{n} x_k^2(x_k-x_{k-1})>C.$$

Solution:

The answer is $\frac{1}{3}$. To see that $C = \frac{1}{3}$ is valid simply observe

$$\sum_{k=1}^{n} x_k^2(x_k - x_{k-1}) = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} x_k^2 \, dx > \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} x^2 \, dx = \int_0^1 x^2 \, dx = \frac{1}{3}.$$

To see that $\frac{1}{3}$ is optimal, consider the sequence $(x_0, x_1, \ldots, x_n) = (\frac{0}{n}, \frac{1}{n}, \ldots, \frac{n}{n})$ as $n \to \infty$.

Alternatively, to show the inequality for $C = \frac{1}{3}$ we may use the estimate $b^2(b-a) > \frac{1}{3}(b^3 - a^3)$ for $b > a > 0$.

Comments:

This of course can be solved with not so big effort without the use of integrals, try it!

Problem 4. Find all functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that $f(f(x) - f(y)) = f(f(x)) - 2x^2 f(y) + f(y^2)$ for all $x, y \in \mathbb{R}$.

Solution:

Let $P(x, y)$ be the assertion $f(f(x) - f(y)) = f(f(x)) - 2x^2 f(y) + f(y^2)$

Let $A = \{x \text{ such that } f(x) = 0\}$ $P(0,0) \implies f(f(0)) = 0$ and so $f(0) \in A$ If $x \in A$, then $P(x, x) \implies f(x^2) = 0$ and so $x^2 \in A$

Let $a \in A$. Then $P(a, x) \implies f(-f(x)) = f(0) - 2a^2 f(x) + f(x^2)$ So, if $a \neq 0$, and setting $a, a^2, a^4, ...$ in the above equation, we get $f(x) = 0$ $\forall x$ which indeed is a solution.

So we'll from now consider $A = \{0\}$ and so $f(x) = 0 \iff x = 0$

$P(x, x) \implies f(f(x)) = 2x^2 f(x) - f(x^2)$ Plugging this in original equation, we get new assertion $Q(x, y)$: $f(f(x) - f(y)) = 2x^2(f(x) - f(y)) + f(y^2) - f(x^2)$

Looking at $Q(x, y)$, we get that $f(x) = f(y) \implies f(x^2) = f(y^2)$ Looking then at $f(f(x)) = 2x^2 f(x) - f(x^2)$, we get that $f(x) = f(y) \implies x^2 = y^2$ and so $y = \pm x$

$P(0, x) \implies f(-f(x)) = f(x^2)$ and, using previous line : $f(x) = \pm x^2$

And so : $\forall x$: either $f(x) = x^2$, either $f(x) = -x^2$

Suppose now that $\exists a \neq 0$ such that $f(a) = a^2$ and $b \neq 0$ such that $f(b) = -b^2$ $P(a, a) \implies f(a^2) = a^4$ $P(b, b)$ $\implies f(-b^2) = -2b^4 - f(b^2)$ and so $f(b^2) = f(-b^2) = -b^4$ $P(a, b) \implies f(a^2 + b^2) = a^4 + 2a^2b^2 - b^4$ And since either $f(a^2 + b^2) = (a^2 + b^2)^2$, either $f(a^2 + b^2) = -(a^2 + b^2)^2$, we get $a = 0$ or $b = 0$, impossible.

And so either $f(x) = x^2$ $\forall x$, either $f(x) = -x^2$ $\forall x$ which both are solutions.

Hence the answer : $f(x) = 0$ $\forall x$ $f(x) = x^2$ $\forall x$ $f(x) = -x^2 0$ $\forall x$

Based on pco's solution on AoPS

Problem 5. A sequence of positive integers $(a_n)_{n \geq 1}$ is of Fibonacci type if it satisfies the recursive relation $a_{n+2} = a_{n+1} + a_n$ for all $n \geq 1$. Is it possible to partition the set of positive integers into an infinite number of Fibonacci type sequences?

Solution:

Define $a_n = \lfloor n\phi \rfloor$ and $b_n = \lfloor n\phi^2 \rfloor$ for all positive integer n, where $\phi = \frac{1+\sqrt{5}}{2}$ is the positive number satisfying $\phi^2 = \phi + 1$.

We claim that for every n, the sequence

$$a_n, b_n, a_{b_n}, b_{b_n}, a_{b_{b_n}}, b_{b_{b_n}}, \ldots$$

is of Fibonacci type. To check this, it suffices to show that $a_{b_n} = a_n + b_n$ and $b_{b_n} = a_{b_n} + b_n$ for all n; however, $b_n = \lfloor n\phi^2 \rfloor = \lfloor n(\phi + 1) \rfloor = a_n + n$ for all n, so the claim $b_{b_n} = a_{b_n} + b_n$ is true.

It suffices to verify that $a_{b_n} = a_n + b_n$ is true, which is equivalent to

$$a_n = a_{b_n} - b_n = \lfloor b_n\phi \rfloor - b_n = \lfloor b_n(\phi - 1) \rfloor = \left\lfloor \frac{b_n}{\phi} \right\rfloor,$$

or

$$a_n = \left\lfloor \frac{a_n + n}{\phi} \right\rfloor.$$

Thus, now we must verify that if $m = \lfloor n\phi \rfloor$, then $m = \left\lfloor \frac{m+n}{\phi} \right\rfloor$.

Suppose that $m = \lfloor n\phi \rfloor$; then, $m < n\phi < m+1$ (equality can't hold since ϕ is irrational). Now, we see that $m < \frac{m+n}{\phi}$ is equivalent to

$$\begin{aligned} m\phi &< m+n \\ m\phi^2 &< m\phi + n\phi \\ m &< n\phi \end{aligned}$$

which is true, and $m+1 > \frac{m+n}{\phi}$ is equivalent to

$$\begin{aligned} (m+1)\phi &> m+n \\ m\phi^2 + \phi^2 &> m\phi + n\phi \\ m + \phi^2 &> n\phi, \end{aligned}$$

which is true since $m+1 > n\phi$ by assumption. Thus, the claim $a_{b_n} = a_n + b_n$ has been verified.

Now, draw an arrow from a_n to b_n, as well as from b_n to a_{b_n}, for each n. By the theory of Beatty sequences, every positive integer n appears in exactly one of $\{a_k\}$ and $\{b_k\}$, so no two arrows point at the same number. In addition, each chain of arrows traces out a Fibonacci type sequence, and every positive integer is in exactly one chain. Finally, we note that since every arrow points from a smaller number to a bigger number, every chain has a number with no incoming arrows (which is the smallest number in the chain).

It suffices to verify that we actually have infinitely many Fibonacci type sequences in our partition; however, this is true since every Fibonacci type sequence has density 0 (since they grow exponentially) while $\mathbb{N}$ has density 1, so $\mathbb{N}$ can't be partitioned into finitely many Fibonacci type sequences.

Based on CantonMathGuy's solution on AoPS

Problem 6. Find all polynomials $P(x) \in \mathbb{R}[x]$ such that $P(x - P(x)) = P(x) - P(P(x))$ for all $x \in \mathbb{R}$.

Solution:

At the beginning let's note that the only constant polynomial satisfying the conditions of the problem is $P(x) = 0$. Furthermore, if x_0 is a root of $P(x)$, then $P(0) = P(x_0 - P(x_0)) = P(x_0) - P(P(x_0)) = 0$, hence the only solutions of degree 1 for the given problem are polynomials of form $P(x) = ax$ for $a \in \mathbb{R}$ (and as one may easily check they indeed satisfy the given equation).

From now on, we may suppose that the degree n of $P(x)$ is greater than 1. Let's first note that $P(x)$ has a repeated root if and only if $P(x)$ and $P'(x)$ have a common root. Consider all polynomials Q_l of the form $Q_l(x) = P(x) - x + l$, where $P(x)$ is a polynomial satisfying the conditions of the problem. Since $Q_l'(x) = P'(x) - 1$ for all $l \in \mathbb{R}$, hence there are only finitely many l's for which Q_l and Q_l' have a common root (indeed, since for every root x_0 of $P'(x) - 1$ there is exactly one value m such that x_0 is a root of $Q_m(x)$). Furthermore, there is at most one $k \in \mathbb{R}$ such that $P(x) - P(k) = (x-k)^n$ (since P is of degree at least 2). Therefore, we may choose k such that $P(x) - P(k) \neq A(x-k)^n$ for any $A \in \mathbb{R}$ and for each l with $P(l) = P(k)$ $Q_l(x)$ has exactly n distinct complex roots.

Let $x_1, x_2, \ldots, x_n$ be roots of $Q_l(x)$, then $k = x_i - P(x_i)$ and thus

$$P(k) = P(x_i - P(x_i)) = P(x_i) - P(P(x_i)) = (x_i - k) + P(x_i - k)$$

so $x_i - k$ are n distinct roots of $Q_{P(k)}(x)$. Since $P(x) - P(k) \neq A(x-k)^n$, there exists l such that $P(l) = P(k)$. Since $Q_l(x)$ has n distinct roots $y_1, y_2, \ldots, y_n$ and for every $i \in \langle 1; n \rangle$ $y_i - l$ is a root of $Q_{P(l)}(x) = Q_{P(k)}(x)$, hence we may order y_i's so that $x_i - k = y_i - l$. Then

$$Q_{P(k)}(x) = Q_{P(l)}(x) = A\prod_{i=1}^{n}(x - y_i) = A\prod_{i=1}^{n}(x - (x_i - k + l)) =$$

$$A\prod_{i=1}^{n}((x + k - l) - x_i) = Q_{P(k)}(x + k - l)$$

Due to assumption, $k \neq l$, hence $Q_{P(k)}(x)$ is a periodic polynomial and hence there exists a such that $Q_{P(k)}(x) - a$ has infinitely many roots, which is clearly a contradiction. Therefore, $P(x) = ax$ is the most general solution to the equation $P(x - P(x)) = P(x) - P(P(x))$.

Problem 7. Let $a_1, a_2, \cdots, a_n \geq 1$ and $A = 1 + a_1 + a_2 + \cdots + a_n$. Define $x_0 = 1$, $x_k = \frac{1}{1 + a_k x_{k-1}}$. Prove that $x_1 + x_2 + \cdots + x_n > \frac{n^2 A}{n^2 + A^2}$

Solution:

Obviously $x_i \leqslant 1$ and because $a_i \geqslant 1$ we have:

$$(a_i - 1)(x_{i-1} - 1) \leqslant 0 \Leftrightarrow 1 + a_i x_{i-1} \leqslant a_i + x_{i-1}$$

Then we conclude that:

$$x_i = \frac{1}{1 + a_i x_{i-1}} \geqslant \frac{1}{a_i + x_{i-1}}$$

Let $X = \sum_{i=1}^{n} x_i$. Using Am-Gm:

$$(A + X)X \geqslant (\sum_{i=1}^{n}(a_i + x_{i-1}))(\sum_{i=1}^{n} x_i) \geqslant (\sum_{i=1}^{n}(a_i + x_{i-1}))(\sum_{i=1}^{n} \frac{1}{a_i + x_{i-1}}) \geqslant n^2$$

Then we have:

$$(A + X)X \geqslant n^2 \Leftrightarrow X \geqslant \frac{-A + \sqrt{A^2 + 4n^2}}{2}$$

We only need to prove:

$$\frac{-A + \sqrt{A^2 + 4n^2}}{2} \geqslant \frac{n^2 A}{n^2 + A^2}$$

Let $t = \frac{A}{n}$, we need to prove:

$$\frac{-t + \sqrt{t^2 + 4}}{2} \geqslant \frac{t}{1 + t^2}$$

After simplification, it is enough to prove:

$$2 + t^2 \geqslant t\sqrt{t^2 + 4}$$

Or:

$$4t^4 \geqslant 0$$

Which is true!

Problem 8. Let n be a positive even integer, and let $c_1, c_2, \ldots, c_{n-1}$ be real numbers satisfying

$$\sum_{i=1}^{n-1} |c_i - 1| < 1.$$

Prove that polynomial

$$P(x) = 2x^n - c_{n-1}x^{n-1} + c_{n-2}x^{n-2} - \ldots - c_1 x + 2$$

has no real roots.

Solution:

Note that $c_i > 0$ for all $i \in \{1, 2, \ldots, n-1\}$. Thus if $x < 0$, clearly P is positive.

We want to prove that for all $x \geq 0$ we have $P(x) > 0$. Note that we can switch x and $\frac{1}{x}$ without switching the problem statement. So without loss of generality assume that $x \leq 1$. Let $a_i = (-1)^i(1 - c_i)$ for all integers $1 \leq i \leq n-1$. Thus the condition becomes $\sum_{i=1}^{n-1} |a_i| < 1$ and our polynomial can be written as $P(x) = x^n + 1 + \frac{x^{n+1}+1}{x+1} - \sum_{i=1}^{n-1} a_i x^i$

Here we have

$$\sum_{i=1}^{n-1} a_i x^i \leq \sum_{i=1}^{n-1} |a_i| < 1 < x^n + 1 + \frac{x^{n+1} + 1}{x + 1}$$

as desired.

pi37 and Wolstenholme - AoPS's users solution

Problem 9. Let $f(x)$ and $g(x)$ be polynomials with non-negative integer coefficients, and let m be the largest coefficient of f. Suppose that there exist natural numbers $a < b$ such that $f(a) = g(a)$ and $f(b) = g(b)$. Show that if $b > m$, then $f = g$.

Solution:

Let $f(x) = \sum c_k x^k$, $g(x) = \sum d_k x^k$. Since $f(b) = g(b)$, $f(x) - g(x) = (x - b)p(x)$, where $p(x)$ is some polynomial with integer coefficients.

Set $p(x) = \sum e_k x^k$. Then $c_k - d_k = e_{k-1} - be_k$. Due to the fact that $c_k, d_k \geq 0$, $c_k - d_k \leqslant c_k < b$. Note that if $e_k < 0$, then $-be_k \geq b > c_k - d_k$ and hence $e_{k-1} < 0$. Therefore, it follows that all the coefficients of p are non-negative (since otherwise we would have $e_{-1} < 0$). Finally.

$$(a - b)p(a) = f(a) - g(a) = 0$$

However, $a - b < 0$ and $p(a) \geq 0$, so $p(a)$ and hence all the coefficients of $p(a)$ are equal to 0. Therefore indeed $f = g$.

based on Alvid - AoPS's user solution

Older division

Problem 1. Find all pairs (a, b) of a real positive numbers a and b such that

$$\lfloor a\lfloor bn\rfloor\rfloor = n-1,$$

for all n positive integer.

Solution:

We have $bn \geq \lfloor bn\rfloor > bn-1$ and so $abn \geq a\lfloor bn\rfloor > abn-a$.

Moreover, $a\lfloor bn\rfloor \geq \lfloor a\lfloor bn\rfloor\rfloor > a\lfloor bn\rfloor - 1$ and so

$$abn \geq \lfloor a\lfloor bn\rfloor\rfloor > abn-a-1$$

Which is $abn \geq n-1 > abn-a-1$ and so

$$ab \geq \frac{n-1}{n} > ab - \frac{a+1}{n}.$$

Setting there $n \to +\infty$, we get $ab = 1$.

Looking now at $\lfloor a\lfloor \frac{n}{a}\rfloor\rfloor = n-1$, we have the equivalent equation $n > a\lfloor \frac{n}{a}\rfloor \geq n-1$ which is also

$$\frac{n}{a} > \lfloor \frac{n}{a}\rfloor \geq \frac{n}{a} - \frac{1}{a}.$$

Left part implies $a \notin \mathbb{Q}$. Right part is $\left\{\frac{n}{a}\right\} \leq \frac{1}{a}$. Since $a \notin \mathbb{Q}$, LHS is dense in $[0,1]$ when n walks around $\mathbb{N}$ so $\frac{1}{a} \geq 1$

Hence the answer

$$\boxed{a \in (0,1) \cap (\mathbb{R} \setminus \mathbb{Q}) \text{ and } b = \frac{1}{a}}$$

Of course above pairs works.

pco - AoPS's user solution

Problem 2. Find all real polynomials $P(x)$ such that for any three integers $a,\ b$ and c satisfying $a+b+c \neq 0$ number

$$\frac{P(a)+P(b)+P(c)}{a+b+c}$$

is an integer.

Solution:

Let $\mathcal{S}$ be the set of integer triplets such that

$$(a,b,c) \in \mathcal{S} \iff a+b+c \neq 0$$

Take $(a,b,c) \in \mathcal{S}$, from the equation we have $\frac{P(a+b)+P(0)+P(c)}{a+b+c}$ is an integer. By the condition of problem we obtain

$$\frac{P(a+b)+P(c)+P(0)}{a+b+c} - \frac{P(a)+P(b)+P(c)}{a+b+c} = \frac{P(a+b)+P(0)-P(a)-P(b)}{a+b+c}$$

is an integer for all $(a,b,c) \in \mathcal{S}$. Taking c large enough, we obtain $P(a+b)+P(0)-P(a)-P(b) = 0$ for all $(a,b) \in \mathbb{Z}^2$.

Notice that

$$P(a+b)+P(0) = P(a)+P(b) \iff P(a+b)-P(0) = (P(a)-P(0))+(P(b)-P(0)),$$

for all $(a,b) \in \mathbb{Z}^2$. Let $Q(x) := P(x)-P(0)$ then $Q(0) = 0$ and

$$Q(a+b) = Q(a)+Q(b)$$

for all $(a,b) \in \mathbb{Z}^2$. Since Q is continuous function, we have $Q(n) = kn$ for $k \in \mathbb{R}$, providing that $P(n) = kn + l$ where $l := P(0)$.

Putting our result into the condition of the problem we see that $l = 0$ and k is an integer. Thus all solutions are in form $f(n) = kn$ and $k \in \mathbb{Z}$.

TTsphn - AoPS's user solution with Mashimaru - AoPS's user improvements

Problem 3. Let k, n be integers such that $1 \leqslant k \leqslant n$,and let $a_1,\ a_2,\ \cdots,\ a_k$ be numbers satisfying the following equations.

$$\begin{cases} a_1 + \cdots\cdots + a_k = n \\ a_1^2 + \cdots\cdots + a_k^2 = n \\ \qquad\vdots \\ a_1^k + \cdots\cdots + a_k^k = n \end{cases}$$

Prove that

$$(x + a_1)(x + a_2)\cdots(x + a_k) = x^k + \binom{n}{1}x^{k-1} + \binom{n}{2}x^{k-2} + \cdots + \binom{n}{k}.$$

Solution:

Let's recall what Newton-Girard equations tell us. Given n numbers $a_1, a_2, \ldots, a_n$, let s_k mean the sum of their k-th powers and τ_k the sum of the products over all their k-tuples. Then

$$s_k - \tau_1 s_{k-1} + \tau_2 s_{k-2} - \ldots + (-1)^k k\tau_k = 0$$

The statement of the problem implies that $s_1 = s_2 = \ldots = s_k = n$. At the same time from Viete formulas we obtain that

$$(x + a_1)(x + a_2)\ldots(x + a_k) = x^k + \tau_1 x^{k-1} + \tau_2 x^{k-2} + \ldots + \tau_k.$$

Therefore, it suffices to prove, using Newton-Girard equations that $\tau_l = \binom{n}{l}$ in this case. We will prove this result using mathematical induction. Let's note that this is indeed true or $l = 1$, since then $s_1 = \tau_1 = n = \binom{n}{1}$. Suppose now that this result is true for all integers m less than some integer l. Then we obtain

$$l\tau_l = s_1\tau_{l-1} - s_2\tau_{l-2} + \ldots + (-1)^{l+1}s_l = n\left(\binom{n}{l-1} - \binom{n}{l-2} + \ldots + (-1)^{l+1}\binom{n}{0}\right)$$

$$\tau_l = \frac{n}{l}\left(\left(\binom{n-1}{l-1} + \binom{n-1}{l-2}\right) - \left(\binom{n-1}{l-2} + \binom{n-1}{l-3}\right) + \ldots + (-1)^{l+1}\binom{n-1}{0}\right)$$

$$\tau_l = \frac{n}{l}\binom{n-1}{l-1} = \binom{n}{l}$$

hence indeed $\tau_l = \binom{n}{l}$ and therefore the problem is solved.

Problem 4. Let a, b, c be positive real numbers, such that $abc = 1$. Show inequality

$$\frac{1}{a^5(b+2c)^2} + \frac{1}{b^5(c+2a)^2} + \frac{1}{c^5(a+2b)^2} \geq \frac{1}{3}.$$

Solution:

Lets make substitution: $x = \frac{1}{a},\ y = \frac{1}{b}$ and $z = \frac{1}{c}$. Now we have to prove that

$$\sum_{cyc} \frac{x^3}{(2y+z)^2} \geq \frac{1}{3}.$$

Using Holder inequality we obtain

$$\left(\sum_{cyc}\left(\frac{x}{(2y+z)^{\frac{2}{3}}}\right)^3\right)^{\frac{1}{3}} \geq \frac{\sum_{cyc} x}{\left(\sum_{cyc}\left((2y+x)^{\frac{2}{3}}\right)^{\frac{3}{2}}\right)^{\frac{2}{3}}} = \frac{\sum_{cyc} x}{(\sum_{cyc} 3x)^{\frac{2}{3}}} = 3^{\frac{3}{2}}\left(\sum_{cyc} x\right)^{\frac{1}{3}}.$$

Hence,

$$\sum_{cyc}\frac{x^3}{(2y+z)^2} \geq \frac{\sum_{cyc} x}{9} \overset{AM-GM}{\geq} \frac{\sqrt[3]{xyz}}{3} = \frac{1}{3}.$$

□

Problem 5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\left(x^2 + xf(y)\right) = xf(x+y)$$

for all reals x, y.

sol As always, let the assertion $f(x^2 + xf(y)) = xf(x+y)$ be denoted by $P(x,y)$.

Assume that $f \not\equiv 0$. Choose $f(z) \neq 0$, and study $P(x, z-x)$ to find $f(x^2 + xf(z-x)) = xf(z)$, where the right side clearly can attain all real values. So f is surjective. Additionally, $P(0,0)$ implies $f(0) = 0$. And $P(x,0)$ gives $f(x^2) = xf(x)$.

Lemma. $f(z) \neq 0 \forall z \neq 0$. Proof: Let S be the set of all reals s with $f(s) = 0$. Clearly $0 \in S$. Now if $s \in S$, then $P(x,s)$ implies $xf(x) = f(x^2) = xf(x+s)$, so $f(x+s) = f(x)$ for all nonzero x. And of course $f(s) = f(0) = 0$, so $f(x+s) = f(x) \forall x \in \mathbb{R}$. Thus all elements of S are periods of f. And all periods p of f satisfy $f(p) = f(0) = 0$, so all periods of f are in S as well; the two sets are precisely the same. This also clearly shows that S is closed under addition and negation.

Next, choose $s \in S$ and look at $P(s,-s)$. We obtain $f(s^2 + sf(-s)) = 0$, so that $s^2 + sf(-s) \in S$. Of course, $s \in S$ gives $-s \in S$ so $f(-s) = 0$ and thus $s^2 \in S$. Thus S is closed under squaring, too.

Now again choose arbitrary $s \in S$, but specify that it be nonzero. Comparing $P(x+s,y), P(x,y)$ gives

$$f(x^2 + xf(y) + 2xs + sf(y) + s^2) = (x+s)f(x+s+y) = (x+s)f(x+y),$$

$$f(x^2 + xf(y)) = xf(x+y).$$

Then whenever $2xs + sf(y) + s^2 \in S$ we must have the left hand sides equal and therefore $sf(x+y) = 0$, which implies $f(x+y) = 0$ and thus $f(x^2 + xf(y)) = 0$ or $x^2 + xf(y) \in S$.

So now for arbitrary nonzero $s \in S$, we have $s^2 \in S$. And for all real x we can find, since f is surjective, y with $f(y) = -2x$ and therefore $2xs + sf(y) + s^2 = s^2 \in S$. Thus $x^2 + xf(y) \in S$ or $-x^2 \in S$. And therefore also $x^2 \in S$. Thus all real numbers are in S, so f is identically zero, contradicting our assumption at the beginning! Of course, what this means is that no nonzero $s \in S$ exist, and therefore $S = \{0\}$, or precisely the result.

Now $P(-f(y), y)$ for nonzero y implies $-f(y)f(y - f(y)) = 0$. But $f(y) \neq 0$ as y is nonzero, so $f(y - f(y)) = 0$. Thus $y - f(y) = 0$ so $f(y) = y$. Therefore $f(x) = x$ for all nonzero x and clearly $f(0) = 0$ also. Thus $f(x) = x \forall x \in \mathbb{R}$, which is a valid solution.

Of course, we could also have the only other possibility - that f is identically zero, also clearly a valid solution.

So $f(x) = 0 \forall x \in \mathbb{R}, f(x) = x \forall x \in \mathbb{R}$ are our two solutions.

Problem 6. Find all triples (x, y, z) of real numbers satisfying the system of equations

$$\begin{cases} x^3 = 3x - 12y + 50, \\ y^3 = 12y + 3z - 2, \\ z^3 = 27z + 27x. \end{cases}$$

Solution:

First of all, notice that triple $(x, y, z) = (2, 4, 6)$ works.

Using first equation see that $x \geq 2 \iff y \leq 4$. Indeed, if $y \geq 4$ then $y \leq 4$ obviously. If $y \leq 4$ we have $x^3 - 3x - 2 \geq 0$ so $(x-2)(x^2 + 2x + 1) \geq 0 \implies x \geq 2$.

Analogously we prove that $z \geq 6 \iff x \geq 2$ and $y \geq 4 \iff z \geq 6$. This follows that only triple satisfying our system of equations is $(2, 4, 6)$. □

Problem 7. Find all polynomials $P(x)$ and $Q(x)$ with real coefficients such that

$$P(Q(x)) = P(x)^{2017}$$

for all real numbers x.

Solution:

It is quite obvious that $deg(Q(x)) = 2017$. let $P(x) = (x-z_1)^{a_1}...(x-z_n)^{a_n}$for some distinct $z_1, ..., z_n$. We have $(Q(x)-z_1)^{a_1}...(Q(x)-z_n)^{a_n} = (x-z_1)^{2017a_1}...(x-z_n)^{2017a_n}$. $x = z_i \implies Q(z_i) = z_j, 1 \leq i \leq n$. if $Q(z) = z_i$ then $x = z \implies z = z_j(*)$. Suppose $Q(z_i) = z_j, Q(z_t) = z_j$, therefore the set $\{Q(z_1), ..., Q(z_n)\}$ has less then $n-1$ elements. this is contradiction since we know there exists some $1 \leq j \leq n$ such that $Q(z_j) = z_i, 1 \leq i \leq n$ using $(*)$. So we have $Q(z_i) \neq Q(z_j) \iff i \neq j.(**)$ Write $Q(x) = H(x) + z_k$ from $(*)$ we get $H(x) = (x-z_1)^{b_1}...(x-z_n)^{b_n}$ so $Q(x) = (x-z_1)^{b_1}...(x-z_n)^{b_n} + z_k$, if $b_i, b_j \neq 0$then we have $Q(z_i) = Q(z_j)$, this is not true because of $(**)$. So we can write $Q(x) = (x-z_i)^{2017} + z_k$.We can do this for all $1 \leq i \leq n$: $Q(x) = (x-z_1)^{2017} + z_{i_1} = (x-z_2)^{2017} + z_{i_2} = ... = (x-z_n)^{2017} + z_{i_n} \implies Q^{'}(x) = 2017(x-z_1)^{2016} = 2017(x-z_2)^{2016} = ... = 2017(x-z_n)^{2016} \implies z_1 = z_2 = ... = z_n \implies P(x) = (x-z)^n, P(x) \in \mathbb{R}[x] \implies z \in \mathbb{R}$ $\boxed{Solution: P(x) = (x-r)^n, Q(x) = (x-r)^{2017} + r}$

Problem 8. Find all polynomials $P(x)$ with real coefficients that satisfy

$$P(x\sqrt{2}) = P(x + \sqrt{1-x^2})$$

for all real x with $|x| \leq 1$.

Solution:

Consider the monic polynomial $f(x)$ of minimal degree that satisfies the equation. Now, assume WLOG that $f(0) = 0$. Letting $x = 0$ we have that $f(1) = 0$. Letting $x = 1$ we have that $f(\sqrt{2}) = 0$. Letting $x = -\frac{\sqrt{2}}{2}$ we have that $f(-1) = 0$. Letting $x = -1$ we have that $f(-\sqrt{2}) = 0$.

Now, differentiating both sides we have that

$$\sqrt{2}f'(x\sqrt{2}) = (1 - \frac{x}{\sqrt{1-x^2}})f'(x + \sqrt{1-x^2}).$$

Letting $x = \frac{\sqrt{2}}{2}$ we have that $f'(1) = 0$. Letting $x = 1$ we have that $f'(0) = 0$. Letting $x = -\frac{\sqrt{2}}{2}$ we have that $f'(-1) = 0$.

This implies that $x^2(x-1)^2(x+1)^2(x^2-2)$ is a factor of f and upon checking we find that this eight degree polynomial (with any real constant added to it) is the minimal solution to the equation.

It is also clear that any polynomial in f is a solution. I claim that these are the only solutions. Consider an arbitrary polynomial in $\mathbb{R}[x]$ that is a solution to this equation. After subtracting off its constant term, it is clear that this polynomial must be a multiple of f or else we could reduce it modulo f and contradict the minimality of f. After subtracting the constant term and dividing by f, we get a new polynomial that also is a solution to the equation. After repeating this process, we inductively obtain the desired result.

Wolstenholme - AoPS's user solution

Problem 9. For each positive integer n, let $f(n)$ be the maximal natural number such that: $2^{f(n)}$ divides $\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2\cdot i+1} 3^i$.

Find all n such that $f(n) = 2017$.

Solution:

Let $a_n = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2\cdot i+1} 3^i$. Then

$$a_n = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2\cdot i+1} 3^i = \frac{(\sqrt{3}+1)^n - (1-\sqrt{3})^n}{2\sqrt{3}}$$

The generating function of this sequence is

$$F(x) = \sum_{i=0}^{\infty} a_n x^n = \frac{1}{2\sqrt{3}}\Big(\sum_{i=0}^{\infty}(\sqrt{3}+1)^n x^n - (1-\sqrt{3})^n x^n\Big) =$$

$$\frac{1}{2\sqrt{3}}\Big(\frac{1}{1-(\sqrt{3}+1)x} - \frac{1}{1-(1-\sqrt{3})x}\Big) = \frac{x}{1-2x-2x^2}$$

Therefore, it follows that our sequence (a_n) is defined by $a_0 = 0, a_1 = 1, a_{n+2} = 2(a_{n+1} + a_n)$.

$$a_n = 2(a_{n-1} + a_{n-2}) = 4(a_{n-2} + 2a_{n-3} + a_{n-4}) = 2^k(a_{n-k} + \binom{k}{1}a_{n-k-1} + \ldots + a_{n-2k})$$

Hence, for all odd n applying the above formula for $k = \frac{n-1}{2}$ implies that we have $f(n) = \frac{n-1}{2}$. Also for numbers, which are of the form $n = 4k+2$ for $k \in \mathbb{Z}$ we have

$$a_n = a_{4k+2} = 2^{2k+1}(a_{3k+2} + \binom{2k+1}{1}a_{3k+1} + \ldots + \binom{2k+1}{1}a_1 + a_0)$$

Since a_1 is the only odd number in this sequence, it follows again that the sum in the brackets is odd and therefore $f(4k+2) = 2k+1$. To solve the problem, it suffices only to find out how does $f(n)$ behaves when $4 \mid n$. However, let's see that

$$a_{2n} = \frac{(\sqrt{3}+1)^{2n} - (1-\sqrt{3})^{2n}}{2\sqrt{3}} = 2^n \frac{(2+\sqrt{3})^n - (2-\sqrt{3})^n}{2\sqrt{3}} = 2^n \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1} 2^{k-2i-1} 3^i$$

Let's note recall that the highest power of two dividing $\binom{n}{k}$ equals to

$$\sum_{i=1}^{\infty} \lfloor \frac{n}{2^i} \rfloor - \lfloor \frac{k}{2^i} \rfloor - \lfloor \frac{n-k}{2^i} \rfloor$$

If k is odd, then clearly, this power of two will be at least the power of two dividing n (since for every power of two dividing n, $\lfloor \frac{n}{2^i} \rfloor$ will gain 1 due to the fact that it is integer and $\lfloor \frac{k}{2^i} \rfloor$ is not.

Therefore, a_{2n} is divible by at least 2 to the power $n + \mathrm{ord}_2(n) + 1$ (where $\mathrm{ord}_2(n)$ denotes the greatest power of 2 dividing n) exactly that much, since only for $i = \frac{n-1}{2}$, the summand is divisible exactly by $2^{\mathrm{ord}_2(n)+1}$. Therefore, $f(4k) = 2k + 2 + \mathrm{ord}_2(k)$. Henceforth, the only n's such that $f(n) = 2017$ are $4024, 4025$ and 4028.

Combinatorics

Younger division

Problem 1. There are 2017 distinct points $A_1, A_2, \ldots, A_{2017}$ given on the Cartesian plane. For any permutation $B_1, B_2, \ldots, B_{2017}$ of $A_1, A_2, \ldots, A_{2017}$ define the shadow of a point P as follows:
Point P is rotated by 180° around B_1 resulting inP_1, point P_1 is rotated by 180° around B_2 resulting in P_2, ..., point P_{2016} is rotated by 180° around B_{2016} resulting in P_{2017}. Then, P_{2017} is called the shadow of P with respect to the permutation $B_1, B_2, \ldots, B_{2017}$. Let N be the number of different shadows of P up to all permutations of $A_1, A_2, \ldots, A_{2017}$. Determine the maximum value of N.

Solution:

Let $(x_1, y_1), (x_2, y_2), \ldots, (x_{2017}, y_{2017})$ be coordinates of $B_1, B_2, \ldots, B_{2017}$, respectively. Note that if (p_1, p_2) are coordinates of point P, then P_1 has coordinates $(2x_1 - p_1, 2y_1 - p_2)$ and similarly P_{2017} has coordinates

$$\big(2(x_{2017} - x_{2016} + \ldots + x_1) - p_1, 2(y_{2017} - y_{2016} + \ldots + y_1) - p_2\big)$$

Therefore, point P can have at most $\binom{2017}{1008}$ shadows, since the parity of indices of B_i's is the only thing that determines the final place of the shadow of P. One might quite easily check that setting $x_i = y_i = 2^i$ results in exactly $\binom{2017}{1008}$ different shadows.

Problem 2. On a table there are $k \geq 2$ piles having $n_1, n_2, \ldots, n_k$ pencils respectively. A move consists of choosing two piles having a and b pencils respectively, $a \geq b$ and transferring b pencils from the first pile to the second one. Find the necessary and sufficient condition for $n_1, n_2, \ldots, n_k$ such that there exists a succession of moves through which all pencils are transferred to the same pile.

Solution:

Let's note at the beginning that we may assume that greatest common divisor of $n_1, n_2, \ldots, n_k$ is 1. If the sum of these n_i's is not a power of 2, then it is divisible by some prime number $p > 2$. Due to our assumption of n_i's being coprime, that would mean that there exists $i \in \{1, 2, \ldots, k\}$ such that $p \nmid n_i$. However, let's see that it is impossible to get rid of all n_i's which are not divisible by p using our operation, since the operation applied to two numbers which are not both divisible by p, leaves us with at least one number not divisible by p. Since at the very end all piles have number of pencils divisible by p, hence we obtain that the sum of n_i's must be a power of 2.

Now let's prove that sum of n_i's being a power of 2 suffices to be sure that we can transfer all the pencils into one pile. We will proceed by induction on l, where 2^l is the sum of n_i's. Case of $l = 0$ clearly works - the only pencil is in one pile. Suppose it is true for $l = m$ ($m \in \mathbb{Z}$, $m \geq 0$) and we want to prove it for $l = m + 1$. Since $m + 1 \geq 1$, the sum of all n_i's is even, so there is even number of odd n_i's and so we can pair them up. After applying the operation on all the pairs, we obtain that all n_i's are even, therefore using the observation that we may assume that n_i's are coprime, we may divide all n_i's by 2 and solve the problem for $l = m$. This gives rise to a sequence of moves transferring all the pencils to one pile. Henceforth, indeed the sum of all n_i's (supposing they have no common factors) being a power of 2 is a necessary and sufficient condition for an existence of moves transferring all the pencils to the same pile.

Problem 3. To n people are to be assigned n different houses. Each person ranks the houses in some order (with no ties). After the assignment is made, it is observed that every other assignment assigns at least one person to a house that person ranked lower than in the given assignment. Prove that at least one person received his/her top choice in the given assignment.

Solution:

Let's take an optimal assignement and let's assume for a contradiction that every person landed in a house that is at least second at his/her list. Let's now draw a bipartite graph with vertices in one part denoting people and in the other the houses. We connect two vertices (person - house) with an edge if and only if this person has been assigned to this house or this house is his/her top choice. The resulting graph has such a property that every person is connected to exactly two houses (since noone lives in their first choice) and every house is connected to at least one person.

Let's make the following path in this graph. Choose randomly some person. Go to his/her top choice. Then let's go from this house to a person which lives in it. Then let's go from this person to his/her top choice, etc.

This process won't terminate, since every house has a person which lives in it and every person has a top choice distinct from the house which he/she lives in. Therefore, using this procedure at some moment we will obtain a cycle. This cycle determines how we should change people's houses cyclicly so that after change every person from the cycle would be in his/her top choice. Hence, we obtained more optimal house assignement, contradicting the assumption about the optimality of our assignement. Thus indeed at least one person received his/her top choice in the given assignment.

Comments:

In fact, you can apply similar reasoning to prove Hall's Marriage Theorem (the statement as well as some exercises you can find here in section 2.: Graph Theory). And once you have proved it and would like some problem that doesn't look that obviously like the variation of Hall's Marriage Theorem, you may try solving this one:

A round-robin tournament among $2n$ teams lasted for $2n-1$ days, as follows. On each day, every team played one game against another team, with one team winning and one team losing in each of the n games. Over the course of the tournament, each team played every other team exactly once. Can one necessarily choose one winning team from each day without choosing any team more than once?

Problem 4. It is known that n is a positive integer, $n \leq 144$. Ten questions of type "Is n smaller than a?" are allowed. Answears are given with delay: The answear to the i'th question is given only after $(i+1)$'th question is asked, $i = 1, 2, \ldots, 9$. The answear to the tenth question is given immediately after it is asked. Find a strategy for identifying n.

Solution:

We will prove by induction on k that in k questions it is possible to determine a number less than F_{k+2}, where F_k denotes k-th Fibonacci number. Let's note that this is indeed true if $k = 1$, since then using only one question we can distinguish between the numbers 1 and 2. Similarly, in case of $k = 2$ $F_4 = 3$ and we can find the searched number in quite an obvious way. Suppose now that this statement is true for all k less than some positive integer $l > 2$.

For brevity, let $P(k)$ denote the statement: "Is n smaller than F_k?". The strategy for $k = l$ is the following (remember that $n \leq F_{l+2}$):
We ask the following questions: $P(l+1), P(l+2), \ldots, P(m)$, where $P(m+1)$ is the first question with the negative answear. Therefore, we know that $F_{m+2} > n \geq F_{m+1}$. Hence, the problem gets reduced to finding $n \leq F_{m+2} - F_{m+1} = F_m$ using only $m-2$ questions. However, this is true by the inductive assumption, so the strategy emerging from this proof indeed allows us to identify $n \leq 144$ in just ten questions.

Problem 5. In a group of several people, some are acquainted with each other and some are not. Every evening, one person invites all of his acquaintances to a party and introduces them to each other. Suppose that after each person has arranged at least one party, some two people are still unacquainted. Prove that they will not be introduced at the next party.

Solution:

We will prove by induction on n - the number of vertices that every connected graph with n vertices becomes a complete graph after the series of meetings as described in the formulation of the problem (that is, every person arranged at least one party). In the case of $n = 2$, the only simple connected graph is at the same time a complete graph, so the result holds here. Now suppose we are given a graph with n vertices and we know that the result holds for all (simple connected) graphs with $m < n$ vertices.

Let's choose the vertex a, which is making the party as the last one. Suppose that deleting it from the graph (as well as edges coming out of it) doesn't disconnect it. By the induction hypothesis we know that before his party all other vertices become connected between each other. Let's assume by a contradiction that by this time a won't be introduced to everyone else. Let W be the set of the people a knows and let V be the set of people a doesn't know (before his party). Denote by b the person in W that is organising the party as the last. Since b is organising the party as the last, he must have got to know a before it and hence a will be on this party. Therefore, it is impossible for b to know any person from V, since this would imply a knowing someone from V. However, since b knows only people from W, who had already organised their parties, it is impossible for him to get to know all the people - contradicting the assumption of graph without a becoming a complete graph. Hence indeed, after all parties except for the last one (if the last vertex doesn't disconnect the graph) the given graph becomes a complete graph.

Now suppose that deleting the last person which organises a party disconnects our graph into several connected

components. From the inductive hypothesis as well as the property derived in the last paragraph we know that before the last party, all connected components will become complete graphs and since these connected components must have been initially connected to the deleted vertex, it follows that the last (a.k.a. deleted) vertex before his party must have got to know absolutely everyone. Thus, the last party makes the graph into a complete graph.

Since we know that every connected graph becomes a complete graph after such series of parties, it follows that two people doesn't know each other after series of parties if and only if they are in two distinct connected components of the graph and hence they will never get to know each other (since there is no path connecting these two vertices).

Problem 6. The numbers $1, 2, \ldots, 2016$ are written around the circumference of a circle. A move consists of exchanging two adjacent numbers. After a sequence of such moves, each number ends up 13 positions to the right of its initial position. lf the numbers $1, 2, \ldots, 2016$ are partitioned into 1008 distinct pairs, then show that in at least one of the moves, the two numbers of one of the pairs were exchanged.

Solution:

Let's make a cut between 1 and 2016 (in the initial position) and instead on a circle let's consider numbers written in a line but with additional move allowed of flipping first and the last number. Using this order, to every permutation of numbers $\{1, 2, \ldots, 2016\}$ we attach numbers $p(1), p(2), \ldots, p(1008)$ - $p(k) = 0$ if numbers from the k-th pair are in the same relative order as in the initial position and $p(k) = 1$ otherwise. Additionally, let $p = p(1) + p(2) + \ldots + p(1008)$.

Assuming that we cannot exchange two numbers from the same pair, it appears that exchanging two consecutive numbers (in our line order) results in absolutely no changes in $p(k)$'s. At the same time, exchanging the first and last number changes exactly two $p(k)$'s. Therefore, we see that the parity of p is invariant under these moves (provided that we don't exchange two numbers from the same pair). Furthermore, let's note that for the permutation, which arises from moving all the numbers 13 positions to the right (but it also works for any odd number), we have an odd p, since from 13 new numbers at the beginning of the line, odd number of them has their pairing partners not in this group and only those pairs have their $p(k)$'s changed. Therefore, indeed it is impossible to move all the numbers 13 positions to the right without exchanging two numbers of one of the pairs.
Comments:

Well, this is not everything that can be taken out from this problem. Do you know what is the sign of permutation? Every permutation can be written as a product of transpositions (which are just permutations switching only two values - such as the ones in the problem - check it yourself) and quite interestingly it appears that no matter how we write it as a product of transpositions (since it is not unique) the number of the transpositions used has always the same parity - therefore, we can talk about odd and even permutations.

Now that you know the solution to the previous problem you may try to prove this result yourself. And once you have solved and started wondering what you can use this knowledge for, you can check out the following problem:

There are n people with hats on the party. Every two of them greets each other exactly once and every greeting consists of exchanging the hats which those people have on their heads at the moment of greeting. Determine all the possible values of n for which there exists an order of greetings such that everyone will have at the end the same hat as at the beginning.

Problem 7. Consider $m + 1$ horizontal and $n + 1$ vertical lines ($m, n \geq 4$) in the plane forming an $m \times n$ table. Consider a closed path on the segments of this table such that it does not intersect itself and also it passes through all $(m - 1)(n - 1)$ interior vertices (each vertex is an intersection point of two lines) and it doesn't pass through any of outer vertices. Suppose A is the number of vertices such that the path passes through them straight forward, B number of the table squares that only their two opposite sides are used in the path, and C number of the table squares that none of their sides is used in the path. Prove that

$$A = B - C + m + n - 1.$$

Solution:

Let A_0, A_1, A_2, A_3 denote respectively the number of table squares with $0, 1, 2, 3$ sides used in the path. Clearly, $A_0 + A_1 + A_2 + A_3 = mn$, since this is the total number of squares and none of them has all four sides occupied

by the path. Since this path makes a cycle on $(m-1)(n-1)$ vertices, hence it has $(m-1)(n-1)$ edges and thus

$$A_1 + 2A_2 + 3A_3 = (m-1)(n-1).$$

Furthermore, there are $(m-1)(n-1)$ interior vertices of the path and exactly $A_2 - B + 2A_3$ turns in it. Therefore,

$$A = (m-1)(n-1) - A_2 + B - 2A_3.$$

Combining all these results together yields

$$A - B + C = ((m-1)(n-1) - A_2 - 2A_3) + (mn - A_1 - A_2 - A_3) =$$

$$(m-1)(n-1) + mn - A_1 - 2A_2 - 3A_3 = mn - (m-1)(n-1) = m + n - 1$$

Hence indeed,

$$A = B - C + m + n - 1.$$

Problem 8. A fly and k spiders are placed in some vertices of three-dimensional lattice $2017 \times 2017 \times 2017$. One move consists of following: firstly, fly goes to some adjacent vertex or stays where it is and then every spider goes to some adjacent vertex or stays where it is (more than one spider can be in the same vertex). Spiders and fly know where are the others all the time. Find the smallest k so that spiders can catch the fly in finite number of moves, regardless of their initial position.

(Vertices in lattice are adjacent if exactly one coordinate of one vertex is different from the same coordinate of the other vertex, and their difference is equal to 1. Spider catches a fly if they are in the same vertex.)

Solution:

Note that we need need at least 2 spiders to catch the fly, since every time fly moves, it has at least 3 possible moves, at most 2 of which are going to be the vertices neighbouring with a single spider. Now we will prove that it is indeed possible to catch the fly using only 2 spiders.

To begin with, let's see that we can make a single spider get at a distance of at most 1 from the fly in each direction. First, let's see that one spider can always win this game on a one dimensional grid - just by going straight to the fly. Hence, just by ignoring the other dimensions a single spider get the same x-coordinate as the fly. Next, let's do the same with the y-coordinate, let just spider play in the y-direction as long as the fly doesn't move in the x-direction, then he adjusts the position. Unfortunately, this strategy doesn't guarantee ever catching the fly, since it can just move all the time in the x-direction forcing spider to do so as well. Therefore, let the spider not move in x-direction until the fly is more than 1 apart from it in this direction. This way, he will gain one additional move in y-direction on every x-direction chase making it get the same y-coordinate as the fly. Similarly, using the same strategy for z-direction, it follows that a spider can get to the fly as near as 1 apart in each direction and also having at least one coordinate the same. Hence, every spider can get to the fly at distance at most 2 in a finite number of moves.

Using the strategy above not only we can make sure that we are exactly two moves apart from the fly but also choose which direction a spider has the same as the fly. Therefore, let's apply this strategy to the two spiders so that one will agree with the fly on y-direction and the other - on z direction. This way, when they get to their positions, the fly will have it moves blocked at least one way in every direction - otherwise, it will get at a distance one from one of the spiders and hence die in the next spiders' move (without loss of generality if fly is on position (x, y, z), then two spiders will be on $(x+1, y, z+1)$ and $(x+1, y+1, z)$ or on $(x+1, y, z+1)$ and $(x-1, y+1, z)$) . If it stays still, we can still move one of the spiders in the x direction, so that it will agree with the fly in this direction. Now, the fly is forced to move, otherwise it dies. On the other hand, doing this move, it is still satisfied that it has one way blocked in each direction. Without loss of generality we may assume that it can only diminish the values in each direction. If spiders do the same moves as fly does, then this property is still satisfied and at some moment the fly will get to the corner of the grid. If it is done with the two spiders being both two apart from the fly, we can get one of them one apart and still cover all the escape routes o the fly. In the next move, the fly dies.

Therefore, the least number of spiders needed to catch the fly is 2.

Problem 9. On a 5×5 board, two players alternately mark numbers on empty cells. The first player always marks 1's, the second 0's. One number is marked per turn, until the board is filled. For each of the nine 3×3 squares the sum of the nine numbers on its cells is computed. Denote by A the maximum of these sums. How large can the first player make A, regardless of the responses of the second player?

Solution:

1	2	3	2	1
2	4	6	4	2
3	6	9	6	3
2	4	6	4	2
1	2	3	2	1

Note the picture on the right side. In each square of 5×5 there is written how many 3×3 squares include this square. If the first player takes the strategy of always picking the largest value he can "gather" during this game 45 "points" out of 81 assuring him that he will have at least 5 1's in every 3×3 square or there will be one square with at least 6 1's in it.

At the same time note that if we pair up the tiles of 5×5 square pairing up tiles from even rows with the ones above them we note that every 3×3 square contains at least 3 such pairing "dominoes". Therefore, if second player uses strategy "if first player plays in some domino respond on the second tile of this domino, otherwise do whatever you wish", it ensures him, A will be lower than 6.

Now comes the tricky part with strategy for first player to always get A to be at least 6. First move goes (quite self-explanatory) to the very middle of the grid. Without loss of generality (by means of rotation) we may assume that the second move is in the bottom two rows. Let's play now in the square just above the middle square. At this moment, three 3×3 squares all contain only two 1's and no 0's at all. The second player is forced now to play above the 1 we've just played if he wants to prevent A from being at least 6 - otherwise one of the top 3×3 squares has still only 2 1's and no 0's - playing in this square all the time wherever is place gives us at least 6 1's irrespectively of the moves of the second player. However, if he does so, we place 1 on the square with 6 points on the right. If he then plays in the top right 3×3 square we win by strategy outlined in the first paragraph - since he won't play the left six, we will gather strictly more than 45 points and hence will have at least one 3×3 square with at least 6 1's. If he plays elsewhere, we play only in the top right 3×3 square getting as before 6 1's.

Therefore, the first player can make A to be 6 regardless of the second player.

Older division

Problem 1. There is a network of roads in a city each road connecting two crossroads such that one can get from any crossroad to any other. We say that this city is then *connected*. This network has an interesting property such that you can split all the roads into two groups A and B such that one can put directions on roads from A so that city is still connected. Similarly, we can put direction on roads from B (leaving roads from A undirected) so that the city is still connected. Prove that we can put directions on all the roads in the city such that this city is still connected.

Solution:

Suppose for a contradiction that this is not true. Let's consider a network of roads which does stand in a contradiction with the statement of the problem and is the one with the fewest number of crossroads (note that it will have strictly more than one crossroad). Let's take an arbitrary road, which, without loss of generality, belongs to group A and which connects crossroads X and Y so that after putting directions on roads from A you can only go by this road from X to Y. Since after putting directions on roads from A, the network is still connected, it is possible to get from Y to X using other roads of the network. It is quite clear that taking the shortest of such paths we don't go through the same crossroad twice, hence we obtain there exists a cycle of crossroads. Let's direct them cyclicly and consider the same problem but with all those crossroads substituted by one crossroad connecting all those roads from "outside" crossroads with one of the merged crossroads - this way we obtain a smaller counterexample. Hence indeed, it is possible to put directions on all the roads in the city such that this city is still connected.

Problem 2. Let A be the set of all permutations $a = (a_1, a_2, \ldots, a_{2017})$ of the 2017 first positive integers such that each permutation satisfies the condition:

there is no proper subset S of the set $\{1, 2, \ldots, 2017\}$ such that $\{a_k | k \in S\} = S$.

For each $a = (a_1, a_2, \ldots, a_{2017}) \in A$, let $d(a) = \sum_{k=1}^{2017} (a_k - k)^2$.

(a) Find the least value of $d(a)$. Denote this least value by d_0.

(b) Find all permutations $a \in A$ such that $d(a) = d_0$.

Solution:

Let's recall that every permutation can be written as a product of disjoint cycles. However, in this case due to the fact that there is no proper subset S of the set $\{1, 2, \ldots, 2017\}$ such that $\{a_k | k \in S\} = S$ it follows that this permutation is just a single cycle. Therefore, if $a = (a_1\, a_2\, \ldots\, a_{2017})$, then

$$d(a) = \sum_{k=1}^{2017} (a_k - a_{k-1})^2$$

where $a_0 = a_{2017}$. Assume without loss of generality that $a_1 = 1$ and $a_k = 2017$. Notice that we can decrease $d(a)$ by sorting $a_2, a_3, \ldots, a_{k-1}$ in the increasing order and $a_{k+1}, a_{k+2}, \ldots, a_{2017}$ in the decreasing order. Therefore, we know that our optimal permutation (the one with the lowest $d(a)$ possible) is a cycle which is monotonous on two arcs of the cycle split by $a_0 = 1$ and $a_k = 2017$.

Hence, the problem reduces to the following problem:

We are given numbers $1, 2, \ldots, 2017$*, where numbers* 1 *and* 2017 *are both considered black and white and the rest of the numbers we colour either white or black. The square length of a sequence* $a_1 < a_2 < \ldots < a_k$ *is defined to be* $(a_2 - a_1)^2 + (a_3 - a_2)^2 + \ldots + (a_k - a_{k-1})^2$*. Find a colouring of numbers from* 2 *to* 2016*, which minimises the sum of the square lengths of both black and white sequence.*

Notice that in the optimal colouring every two consecutive squares are of different colours. Here is why: Note that colouring of all the numbers with one colour is clearly not optimal. Therefore, every monochromatic sequence of consecutive numbers, without loss of generality white numbers, borders with at least one black number (which is neither 1 nor 2017). Suppose a is black and $a + 1, a + 2, \ldots, a + k$ are white. Flipping colours of all numbers from 2 to $a + 1$ results in the sum of square lengths diminishing by $(k + 1)^2 + 1^2 - k^2 - 2^2 = 2(k - 1)$ (since monochromatic pairs $(a, a + k + 1)$, $(a + 1, a + 2)$ disappeared giving rise to $(a + 1, a + k + 1)$, $(a, a + 2)$).

Therefore, indeed every two consecutive numbers have to be of different colours. There are only two such colourings - each of which consists of colouring all odd numbers with one colour and all the even with the second. Getting back to original problem the least value of $d(a)$ possible equals

$$1^2 + 1007 \cdot 2^2 + 1^2 + 1008 \cdot 2^2 = 4 \cdot 2015 + 2 = 8062$$

and one can obtain it only from permutations (1 2 4 6 ... 2016 2017 2015 2013 ... 5 3) and (1 3 5 ... 2015 2017 2016 2014 2012 ... 4 2).

Problem 3. *Manhattan distance* between two points (a, b) and (c, d) is defined to be $|a - c| + |b - d|$. Suppose only two distances occur between all pairs of distinct points of some point set. What is the maximal number of points in such a set?

Solution:

At the beginning we will need one small lemma:

Lemma (Erdős–Szekeres theorem):
Sequence of $n^2 + 1$ real numbers contains either a nonincreasing or a nondecreasing subsequence of length $n + 1$.

Proof:
Let's attach to every number in our sequence a pair of numbers (a, b) such that a denotes the length of the longest nonincreasing subsequence ending on this number and b - the length of the longest nondecreasing subsequence ending on this number. Let's see that no two numbers in our sequence m, n can be given the same pair of numbers (a, b). Indeed, if m is further in the sequence than n, then if $m \geq n$, then we can form a nondecreasing sequence of length $a + 1$ ending on m by just taking the sequence of length a ending on n and adding to it m. Similarly we handle the case of $m < n$.

Let's note that it is possible to construct a set of 9 points such that only two distances occur between all pairs of distinct points of this set. Indeed, it just suffices to take point $(0, 0)$ and all lattice points, which are 2 apart from the origin in the Manhattan distance.

To prove that this is indeed the maximal number of points possible we will use our lemma. Suppose for a contradiction that there exists a set of at least 10 points having only two distances among all their mutual Manhattan distances. Going from the points with smallest x-coordinate to the ones with the largest x-coordinate, we right down a sequence of their y-coordinates. According to our lemma, there exists a nondecreasing or nonincreasing subsequence of this sequence of length at least 4. Taking the corresponding points and looking at the differences from the first point to the three other yields three different Manhattan distances.

Problem 4. To each vertex of a regular pentagon an integer is assigned, so that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers x, y, z respectively, and $y < 0$, then the following operation is allowed: x, y, z are replaced by $x + y, -y, z + y$ respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps.

Solution:

Consider x_i for $i = 1, 2, 3, 4, 5$ as the five numbers in the pentagon. Now, consider $x_4 = y < 0$, and by the problem hypothesis $S = \sum_{i=1}^{n} x_i > 0$, we have by the function $f(x_1, x_2, x_3, x_4, x_5) = \sum_{i=1}^{n} (x_{i+2} - x_i)^2$ where x_6 can be thought of as x_1 and x_7 as x_2. Obviously, if we consider f_2 as the new value of $f(x_i, i = 1, 2, 3, ..5)$ after the transformation and f_1 as the initial one , we get that $f_2 - f_1 = 2sx < 0$ by the given condition..... So, the function is decreasing on the positive integers , by the Principle of Finiteness of a Decreasing Sequence on positive integers, we get that the operation has to stop at some point.

Problem 5. Alice and Bob play the following game. To start, Alice arranges the numbers $1, 2, \ldots, n$ in some order in a row and then Bob chooses one of the numbers and places a pebble on it. A player's turn consists of picking up and placing the pebble on an adjacent number under the restriction that the pebble can be placed on the number k at most k times. The two players alternate taking turns beginning with Alice. The first player who cannot make a move loses. For each positive integer n, determine who has a winning strategy.

Solution:

We will show that Alice has a winning strategy if and only if

$$2 \mid 1 + 2 + \ldots + n = \frac{n(n+1)}{2} \qquad 4 \mid n(n+1) \qquad n \equiv 0 \text{ or } 3 \pmod 4$$

It is easy to see that if $n \equiv 0$ or $3 \pmod 4$, then by using permutation $1\,3\,5 \ldots n-1\,n$ $n-2 \ldots 4\,2$ in the case of $4 \mid n$ or $1\,3\,5 \ldots n-2\,n\,n-1\,n-3 \ldots 4\,2$ in the latter case, allows Alice to pair up all the pebbles in such a way that pebbles from every pair are on neighbouring places in a row. Therefore, her

strategy is to always take a pebble which is paired up with one that Bob has just taken.

In the case of $\frac{n(n+1)}{2}$ being odd, we just greedily try to pair up the pebbles. Hence, we pair up all pebbles from the first pile with the ones from the second pile. Then we pair up all the unpaired pebbles from the second pile with the ones from the third pile. We proceed this way, however since it is impossible to pair up all the pebbles (since their number is odd), there exists $k \in \mathbb{Z}$ such that the number of unpaired pebbles from the k-th pile (unpaired after pairing pebbles from $k-1$-st pile) will be greater than the number of pebbles in the $k+1$-st pile. Bob strategy is as follows: he first takes pebble from k-th pile. From now on, he plays Alice's strategy for the board, where he ignores all the piles from $k+2$ to n and some additional pebbles on k-th pile, which is a board with all pebbles paired up as in the first case.

Let's note that he can indeed ignore piles from $k+2$ to n, since his pairing strategy makes him go to k-th pile, whenever Alice takes a pebble from $k+1$-st pile. Also, Alice will always take pebbles from the piles, which are at odd distance from k, hence she will never take advantage of the additional pebbles lying there. Therefore, indeed Alice has a wining strategy if and only if $4 \mid n(n+1)$.

Problem 6. Let $T = \{1, 2, 3, \ldots, 14, 15\}$. Say that a subset S of T is handy if the sum of all the elements of S is a multiple of 5. For example, the empty set is handy (because its sum is 0) and T itself is handy (because its sum is 120). Compute the number of handy subsets of T.

Solution:

Let

$$F(x) = \sum_{k=0}^{120} a_k x^k$$

where a_k denotes number of subsets of T with the sum of all its elements equal to k (here we assume the sum of the elements of the empty set equals 0). Let's note that considering all the possibilities of the form: k can be in the set (changing its sum by k, which in the form above is equivalent to multiplying by x^k) or not, we obtain that

$$F(x) = (1+x) \cdot (1+x^2) \cdot \ldots \cdot (1+x^{15})$$

If A, B, C, D, E are respectively the numbers of the subsets of T with sums being $0, 1, 2, 3, 4$ modulo 5, then $F(\omega) = A + B\omega + C\omega^2 + D\omega^3 + E\omega^4$, where ω denotes the fifth root of unity.

$$F(\omega) = (1+\omega) \cdot (1+\omega^2) \cdot \ldots \cdot (1+\omega^{15}) = \left((1+\omega) \cdot (1+\omega^2) \cdot \ldots \cdot (1+\omega^5)\right)^3$$

Let's recall however that

$$x^5 - 1 = (x-\omega) \cdot (x-\omega^2) \cdot \ldots \cdot (x-\omega^5)$$

$$(1+\omega) \cdot (1+\omega^2) \cdot \ldots \cdot (1+\omega^5) = (-1)^5((-1)-\omega) \cdot ((-1)-\omega^2) \cdot \ldots \cdot ((-1)-\omega^5) = (-1) \cdot ((-1)^5 - 1) = 2$$

Therefore, $A + B\omega + C\omega^2 + D\omega^3 + E\omega^4 = 2^3 = 8$. At the same time, $A + B + C + D + E = F(1) = 2^{2015}$, hence $5A - 32 = 2^{2015}$, so $A = \frac{2^{2015} + 32}{5} = 6560$.

Problem 7. Paweł has put the participants of MBL in three lecture rooms. He is only allowed to transfer people from one room to another so that the number of participants in the latter is doubled.

(a) Prove that Paweł can always empty one of the rooms.

(b) Can he put all the participants in one room?

Solution:

At the beginning let's note that it is impossible to move all the participants to one room if the number of participants is odd. To prove that he can empty one of the rooms it suffices to prove that there exists a sequence of moves such that after applying them, the minimum of the three numbers (of participants in the rooms) is strictly less than previously. Here it is: Suppose that there a, b, c people in the three rooms, respectively, and that $a \leq b \leq c$. Let's write $b = aq + r$, where $0 \leq r < a$. In every move, we will double the number of people in a's room from $2^l a$ to $2^{l+1} a$, pairing it in the allowed operation with b if q has 1 in binary system on l-th place and with c otherwise.

Note that using this operation $\lfloor \log_2 q \rfloor + 1$ times we will reduce b by aq getting a triple of numbers with minimum being not greater than $q < a$. Therefore, we will get a triple with strictly lower minimum. One may wonder whether indeed at every step of this operation the first room will be the one with the lowest number of

participant in the chosen pair - it is clear in the case of pairing with the second room, but also in the case of pairing with the third one it is not that hard. The last operation will be done between the first and the second room, hence due to properties of this sequence of moves c will lose strictly fewer participants than b. Combining it with the fact that $b < c$ gives the required result. Going back to the strategy, since we obtained a sequence of moves which diminished the minimum of the triple of these numbers, hence at some moment we will get to the point where one of the rooms will be empty, which ends the proof.

Problem 8. A country has 2017 cities. Every pair of cities is connected by a road. Each road is owned by one of 10 companies. Prove that there must be a way to travel in a circuit of odd length along a sequence of roads that are all owned by a single company.

Solution:

Let's assume for a contradiction that a complete graph K_{2017} can be decomposed into 10 subgraphs - each having all its cycles of even length. However, this property is satisfied only by bipartite graphs. Indeed, every bipartite graph has all its cycles of even length (otherwise it would end the cycle not only in the same vertex as it started but not even the same part of bipartite graph) and all graphs with that property are bipartite graphs (consider colouring greedily the vertices of the graph with 2 colours such that neighbouring vertices would have different colours - it cannot fail since there is no odd cycle).

Suppose now we want to colour the vertices of our complete graph in such a way that any two neighbouring vertices have different colours. How many colours do we need? At least 2017, of course. On the other hand, however, if it was possible to decompose this graph into 10 bipartite graphs (such that every edge would be in some bipartite graph), then it would be possible to colour this graph with only $2^{10} = 1024 < 2016$ colours - since every bipartite graph is 2-colourable and the composition of this 10 graphs would give 2^{10}-colourable graph with the colours determined by all the combinations of colours in given bipartite graphs - since every 2 vertices are connected in some bipartite graph, their final colours would also differ. Therefore indeed there exists a circuit of odd length of roads that are all owned by a single company.

Problem 9. Suppose that each of the vertices of a simple graph is equipped with an indicator light and bubble. Each vertex's button simultaneously toggles the states of all its neighbours as well as its own state. Initially all lights are off. Prove that it is possible to turn on all the lights.

Solution:

Let $\mathbf{A}$ be the sum of the identity matrix and adjacency matrix of this graph. The problem requires to prove that there exists a solution to the equation $\mathbf{Ax} = \mathbf{1}$ (which takes place $\mathbb{F}_2$).

Lemma: Matrix equation $\mathbf{Ax} = \mathbf{1}$ in $\mathbb{F}_2$ has no solutions if and only if there is an odd number of rows in $\mathbf{A}$ that sum up to $\mathbf{0}$.

Proof: Note that the if direcion is pretty straight-forward. When restricted only to those rows and summed up, equation $\mathbf{Ax} = \mathbf{1}$ yields $\mathbf{0x} = \mathbf{1}$ having no soluions. On other hand, if there is no such collection of rows, then we may just apply Gaussian elimination $[\mathbf{A}, \mathbf{1}]$ to row-reduced-echelon form. Since in $\mathbb{F}_2$ linear combinations are just sums of the rows, the mentioned condition assures us that we won't end up looking like $[0, 0, \ldots, 0, 1]$ leading to a valid solution.

Suppose now for a contradiction that $\mathbf{Ax} = \mathbf{1}$ indeed has no solution. Then due to the lemma there exists an odd collection of rows adding up to $\mathbf{0}$. If $\{r_1, r_2, \ldots, r_t\}$ is such a collection of rows, then consider a graph having as an adjacency matrix $\mathbf{A}$ reduced to these rows and corresponding columns, but ignoring 1's on the diagonal. Since the sum of ones in every row (except for the diagonal element) equals the degree of a vertex, it follows that we obtain a graph with odd number of vertices, each of which has an odd degree (because we ignored the 1's on the diagonal). We reached a contradiction, therefore indeed we can turn on all the lights.

Geometry

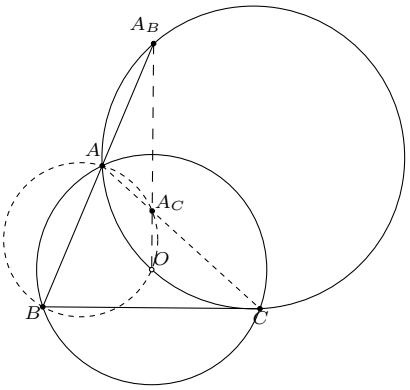
Younger division

Problem 1. Denote by O the circumcenter of $\triangle ABC$. The circumcircle of $\triangle AOC$ and $\triangle AOB$ intersects the lines AB and AC at points $A_B,\ A_C$ respectively. Similarly define points $B_A,\ B_C,\ C_A,\ C_B$. Prove that lines $A_BA_C,\ B_AB_C,\ C_AC_B$ have a common point.

Solution:

We prove that line A_BA_C passes through O.

Consider an inversion in the circumcircle of $\triangle ABC$. This transformation maps the circumcircle of $\triangle ACO$ to the line AC and maps the line AB to the circumcircle of $\triangle ABO$. Therefore, the point A_B as the intersection point of the line AB and the circumcircle of $\triangle ACO$ is mapped into intersection point of the line AC and the circumcircle of $\triangle ABO$ so A_C. Therefore, points $O,\ A_B,\ A_C$ are collinear.



Analogously we prove that B_AB_C and C_AC_B go through the point O.

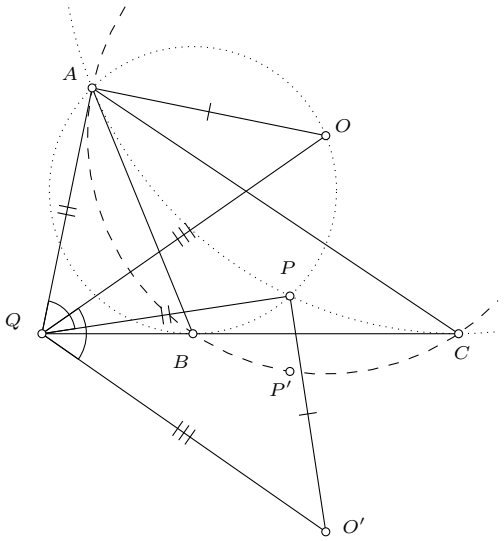
□

Hint: Try inversion.

Problem 2. Let ABC be an acute scalene triangle with O as its circumcenter. Point P lies inside triangle ABC and satisfies $\sphericalangle PAB = \sphericalangle PBC$ and $\sphericalangle PAC = \sphericalangle PCB$. Point Q lies on line BC and satisfies $QA = QP$. Show that

$$\sphericalangle AQP = 2\sphericalangle OQB.$$

Solution:
Denote by P' and O' reflections of the points P and O with respect the line BC respectively. We have to prove that $\sphericalangle O'QO = \sphericalangle PQA$ so it is enough to prove congruency of $\triangle QO'P$ and $\triangle QOA$.



Since $\sphericalangle PCB = \sphericalangle PAC$ and $\sphericalangle PBC = \sphericalangle PAB$ we have

$$\sphericalangle BP'C = \sphericalangle BPC = 180^\circ - \sphericalangle BAC.$$

Hence, the point P' lies on circumcircle of $\triangle ABC$ so $PO' = P'O = OA$. Moreover, by symmetry $QO = QO'$ and $QP = QA$ so $\triangle AQO \equiv \triangle QPO'$ indeed.

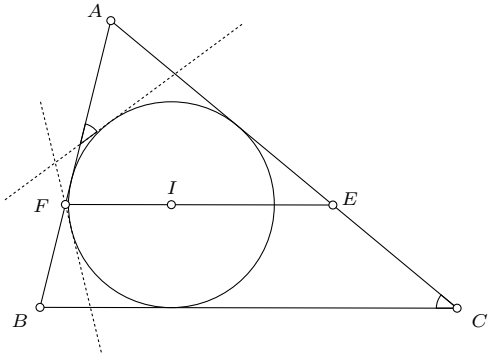
Reader could have noticed useless dotted circles in the upper picture. We have added them to show that this configuration isn't about triangle ABC actually but two circles tangent to the given line. Furthermore, it can be proven that the lines QA and AP' are tangent to the circumcircle of triangle ABC. Hence, $ABP'C$ is a harmonic quadrilateral. □

Problem 3. Let ω be the incircle of the $\triangle ABC$ with center I. The line which contains I and is parallel to the line BC intersects the sides AC and AB at the points E and F respectively. Let E' be the symmetric point of E through BI. Let F' be the symmetric point of F through CI. Prove that the line $E'F'$ is tangent to the circle ω.

Solution:
Let $\mathbf{S}_\ell$ be the symmetry with respect to line ℓ .

Denote by $\tau_B = \mathbf{S}_{CI}(\mathbf{S}_{EF}(AB))$ and $\tau_C = \mathbf{S}_{BI}(\mathbf{S}_{EF}(AC))$. These lines are tangent to ω. As angle between images is equal double angle between axes of symmetry we have $\measuredangle(\tau_B, AB) = 2\measuredangle(CI, EF) = \measuredangle ACB$ and $\measuredangle(AC, \tau_C) = 2\measuredangle(EF, BI) = \measuredangle CBA$ so $\tau_B \equiv \tau_C$.

Now it is enough to notice that points E' and F' lie on tangent τ_B.



□

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Problem 4. Two distinct circle ω_1 and ω_2 meet at two distinct points P and Q. A line ℓ intersects ω_1 at the points $A,\ C$ and intersects ω_2 at the points $B,\ D$ such that the points A, B, C and D are all distinct and lie on ℓ in this order. Let T be a point on the line PQ such that P lies between T and Q. Lines $AT,\ BP$ intersect at X and lines $DT,\ CP$ intersect at Y. Finally, let M and N be the midpoints of the segments AD and BC respectively. Prove that $TM,\ PN$ and XY are concurrent.

Solution:

Let TA intersects ω_1 at the point E and let TB intersects ω_2 at the point F. Since T lies on the radical axis of the two circles, quadrilateral $AEFB$ is cyclic. As

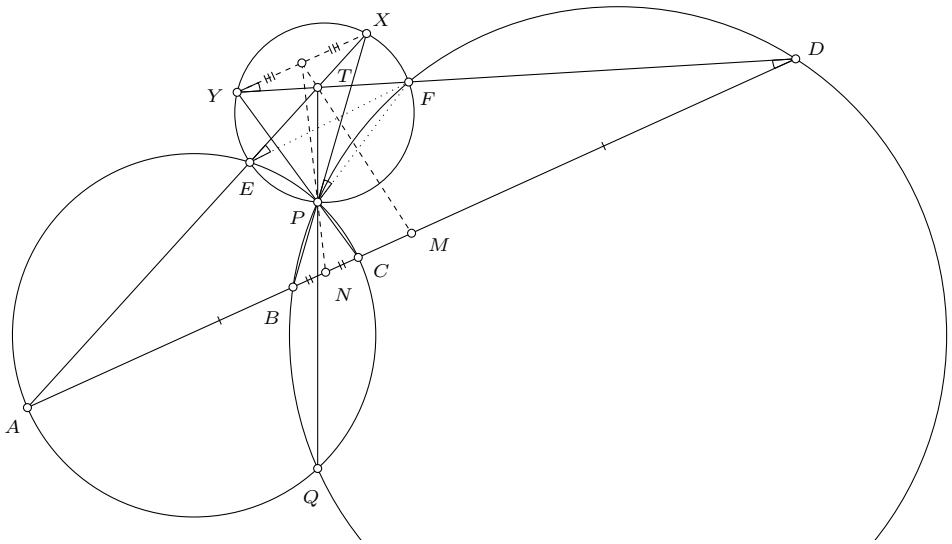
$$\measuredangle XEF = \measuredangle TDA = \measuredangle XPF,$$

quadrilateral $XEPF$ is also cyclic.

Similarly, quadrilateral $YEPF$ is cyclic thus $XYEPF$ is cyclic. Hence,

$$\measuredangle TYX = \measuredangle FET = \measuredangle TDA$$

so $XY \| AD$. Since M and N are the midpoints of AD and BC respectively, lines MT and NP concur at the midpoint of the segment XY.



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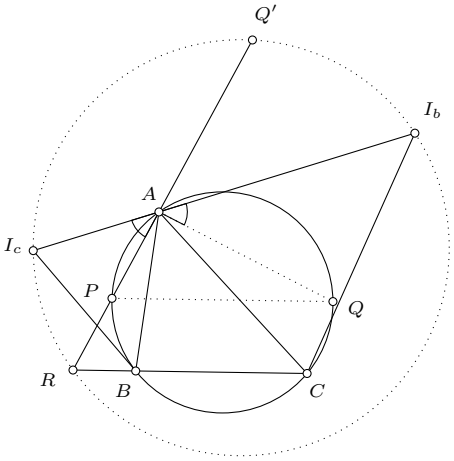
Problem 5. (Kürschák 2012, problem 1) Let I_b and I_c be the B-excenter and C-excenter of $\triangle ABC$ respectively. Consider a chord PQ of circumcircle of $\triangle ABC$ which is parallel to BC and intersects the segments AB and AC. Suppose the lines BC and AP intersect at R, prove that

$$\measuredangle I_b Q I_c + \measuredangle I_b R I_c = 180^\circ.$$

Solution:

First of all, notice that $\triangle AI_cB \sim \triangle AI_bC$. Denote by Q' reflection of the point Q about the line I_bI_c. As $PQ \parallel BC$ we have $\sphericalangle PAB = \sphericalangle CAQ$. Taking together similarity and equality of angles we obtain $\sphericalangle I_cAP = \sphericalangle QAI_b$ so Q' lies on the line AP.



Consider transformation Φ - composition of the inversion about circle centered at A with radius $\sqrt{AB \cdot AC}$ and the symmetry about the inner bisector of $\sphericalangle BAC$. Notice that $\Phi(B) = C$, so circumcircle of $\triangle ABC$ is transformed into line BC under Φ. Line AP is mapped to line AQ so $\Phi(Q) = R$.

Upper results give conclusion that $AQ \cdot AR = AB \cdot AC$. Using similarity of the triangles pointed earlier we get

$$AI_c \cdot AI_b = AB \cdot AC = AQ \cdot AR = AQ' \cdot AR$$

so I_cRI_bQ' is cyclic.

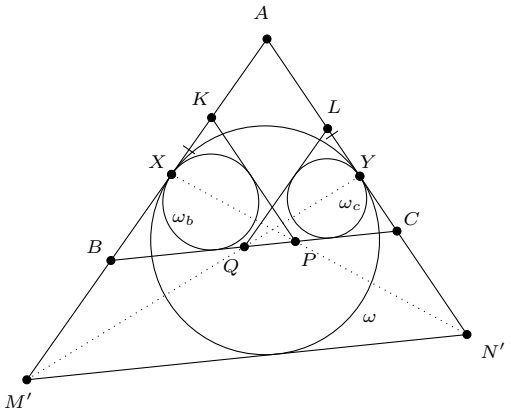
Therefore,

$$\sphericalangle I_bQI_c + \sphericalangle I_bRI_c = \sphericalangle I_bQ'I_c + \sphericalangle I_bRI_c = 180^\circ$$

what we had to prove.

Problem 6. On the sides AB and AC of $\triangle ABC$ choose points K and L respectively. Let P and Q be the points laying on the side BC such that $KP \parallel AC$ and $LQ \parallel AB$. The incircle of $\triangle BKP$ touches side AB at the point X. The incircle of $\triangle CLQ$ touches side AC at the point Y. The lines QY and AB intersect at the point M, the lines XP and AC intersect at the point N. Show that if $AX = AY$ then $MN \parallel BC$.

Solution:
Consider circle ω tangent to the lines AB and AC at the points X and Y respectively. Let M' and N' be the points lying on AB and AC respectively such that $M'N' \parallel BC$. Denote by ω_b and ω_c incircle of $\triangle BPK$ and $\triangle CQL$ respectively.



Since $KP \parallel AN'$ and $BP \parallel M'N'$ there exist homothety with center lying on the line KB and mapping $\triangle KPB$ onto $\triangle AM'N'$. As incircle of those triangles are tangent at the point X, point X is the center of the considered homothety. Therefore, points X, P and N' are collinear so $N' = N$. Analogously we prove that $M' = M$ so the line MN is parallel to the line BC.

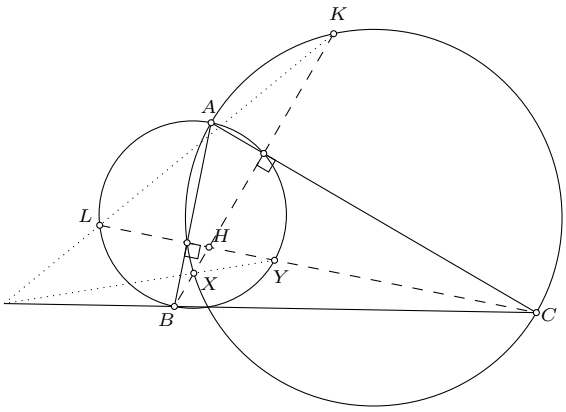
Problem 7. Triangle ABC is given. The line perpendicular to AC and going and through B intersects the circle with diameter AC at the points X and K where X lies closer to B than K. Analogously the line perpendicular to AB and going through C intersects the circle with diameter AB at the points Y and L where Y lies closer to C than L. Prove that the intersection point of XY and KL lies on the line BC.

Solution:

First of all, denote by ω the circle with diameter AC. Let H be the orthocenter of triangle ABC.
We can see that the polar of B with respect to ω passes through the point H. Hence, by well know lemma $(B, H; X, K) = -1$. Analogously we can prove that $(C, H; Y, L) = -1$.
Let $T := KL \cap XY$, as

$$T(B, H; X, K) = -1 = T(C, H; Y, L)$$

points $T,\ B$ and C are collinear what we had to prove. $\square$



Problem 8. Let ω be the incircle of $\triangle ABC$ such that $AB = AC$. The circle ω touches sides $BC,\ CA$ and AB at the points $D,\ E$ and F respectively. The line BE intersects ω again at the point X, the line CF intersects ω again at the point Y. Line XY intersects segments $AB,\ AC,\ DF$ and DE at the points $K,\ L,\ P$ and Q respectively. Show that

$$KX = XP = PQ = QY = YL.$$

Solution:
First of all, quadrilateral $DEFX$ is harmonic so

$$-1 = D(X, E; D, F) = D(X, Q; P, B)$$

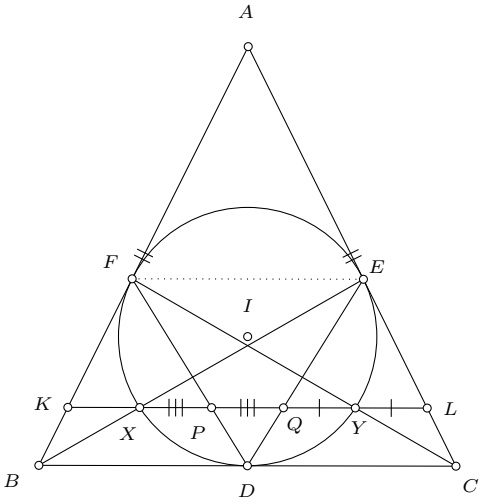
hence P is the midpoint of the segment XQ.

On the other hand, quadrilateral $DYEF$ is harmonic so

$$-1 = E(E, D; Y, F) = E(L, Q; Y, F).$$

By symmetry $EF \| BC$ so Y is the midpoint of the segment QL.

Using symmetry we obtain our thesis.



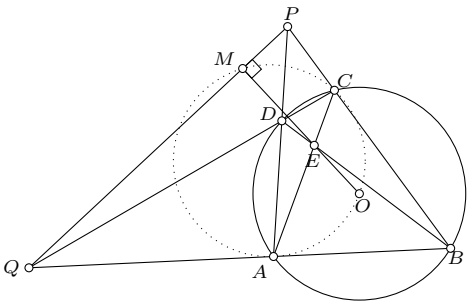
Problem 9. Let ABC be an acute scalene triangle inscribed in the circle ω. Circle ω, centred at the point O, passes through the points B and C and intersects the sides AB and AC at the points E and D respectively. The point P lies on the arc BC of circumcircle of $\triangle ABC$ with the point A. Prove that the lines $BD,\ CE$ and OP are concurrent if and only if $\triangle PBD$ and $\triangle PCE$ have the same incenter.

Solution:
First of all, we will introduce useful lemma.

Lemma: Let $ABCD$ be the quadrilateral inscribed in the circle centred at O. Segments AC and BD intersect at the point E. The lines AD and BC intersects at the point P and the lines AB and CD intersects at the point Q. The line OE meets the line PQ at the point M. the point M is the Miquel point of quadrilateral $ABCD$.

Denote by o circumcircle of quadrilateral $ABCD$. Since the point E lies on the polar of the points P and Q with respect o, the line PQ is the polar of the point E with respect o.

Consider an inversion in circle o. From following observations we conclude that our inversion maps E to M. The points $A,\ E$ and C are collinear so the point $M,\ A,\ O$ and C lies on the one circle. Therefore, $\sphericalangle EMC = \sphericalangle OAC$ and analogously $\sphericalangle DME = \sphericalangle OBD$.

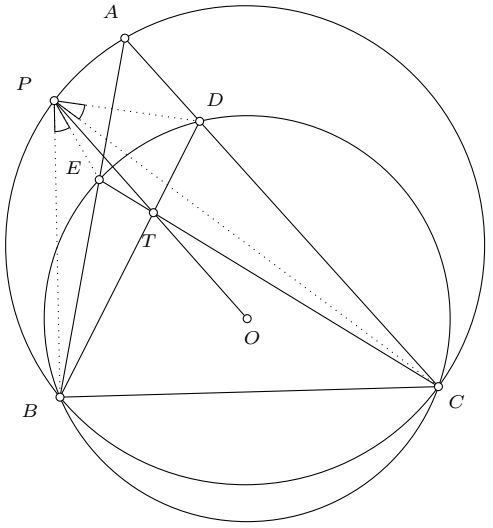


Now, we can easily calculate

$$\sphericalangle DMC = \sphericalangle DME + \sphericalangle EMC = \sphericalangle OBD + \sphericalangle OAC = \sphericalangle APB$$

so the quadrilateral $MDCP$ is concyclic. Similarly we prove $MABP$ is concyclic and it finishes our proof.

Lets come back to the main problem



Suppose OP, BD and CE intersect at the point T. Using lemma the point P is the Miquel point of quadrilateral $BCDE$. Since the quadrilateral $PBOD$ is cyclic and $OD = OB$, the line PO is the bisector of $\sphericalangle BPD$. Applying trillium theorem, intersection point of the line PO and circle ω is the incenter of $\triangle BDP$. Analogously this point is the incenter of $\triangle PEC$.

Suppose $\triangle PBD$ and $\triangle PEC$ share the same incenter. In this case $\sphericalangle BPD$ and $\sphericalangle CPE$ share the same bisector so

$$\sphericalangle BPE = \sphericalangle CPD.$$

Since $\sphericalangle PBE = \sphericalangle PCD$, we obtain that $\triangle PBE$ and $\triangle PCD$ are directly similar. It implies that the point P is the Miquel point of the quadrilateral $BCDE$.

□

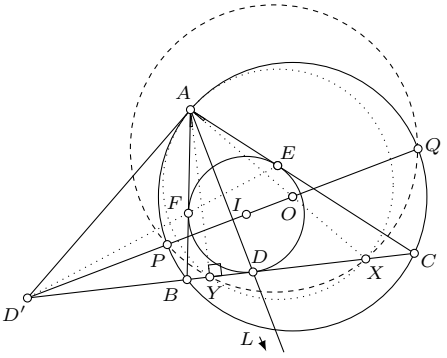
Older division

Problem 1. In acute $\triangle ABC$, let o be its circumcircle and ω be its incircle. Denote by O and I center of o and ω respectively. Tangents at the points B and C to o meet at the point L. The circle ω touches the side BC at the point D. The line AY is perpendicular to BC at the point Y and AO meets BC at the point X. Suppose the line OI meets o at the points P and Q. Prove that if $A,\ D$ and L are collinear, then the points $P,\ Q,\ X$ and Y are concyclic.

Solution:
First of all, notice that circumcircle of $\triangle AXY$ is tangent to o. Let D' be the point lying on the line BC such that $(D, D'; B, C) = -1$. As AD is the symmedian, AD' is tangent to o. Since $D',\ E$ and F are collinear, the point D' is the pole of AD with respect to ω and of AL with respect of o. Hence, $D \in OI$. Using power of point

$$D'P \cdot D'Q = D'A^2 = D'Y \cdot D'X$$

so the quadrilateral $PYXQ$ is cyclic.



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Problem 2. Let O and I be the circumcenter and incenter of triangle ABC respectively. The perpendicular from I to OI meets BC and the external bisector of angle A at points Y and X respectively. In what ratio does I divide the segment XY?

Solution:

Let $\triangle I_aI_bI_c$ be the excentral triangle of $\triangle ABC$ and $\overline{A'B'C'}$ be the perspective axis of $\triangle ABC$, $\triangle I_aI_bI_c$. Denote by Z the intersection point of lines AI_A and XY

As $I_B,\ I_C,\ B$ and C lies on the one circle so the point A' is the radical center of circumcircle of quadrilateral $I_BI_CBC,\ \triangle ABC$ and $\triangle I_AI_BI_C$. It implies that the point A' lies on radical axis of circumcircles of $\triangle ABC$ and $\triangle I_AI_BI_C$. Analogously we prove that the points $B',\ C'$ also lie on this axis. Hence, $\overline{A'B'C'} \perp OI \perp XY$ so $XY \parallel \overline{A'B'C'}$

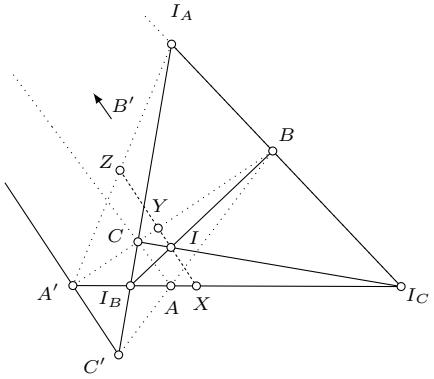
Since

$$-1 = (B', B; I_A, I_C) = A'(B', Y; Z, X) = (\infty, Y; Z, X),$$

point Y is the midpoint of segment ZX. On the other hand,

$$-1 = A'(I_A, I; B, A) = (Z, I; Y, X)$$

so we conclude $IX = 2IY$.



□

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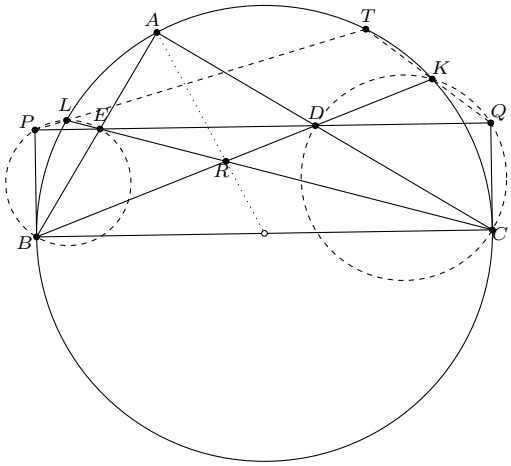
Problem 3. Denote by τ the circumcircle of $\triangle ABC$. Let the points D and E lie on the sides AC and AB respectively. Let $K := BD \cap \tau$ and $L := CE \cap \tau$. Tangents to τ at the points B and C intersect the line DE at the points P and Q respectively. Prove that PL and QK intersect on τ.

Solution:

Let $R = BD \cap CE$. Take a projective transformation which preserves τ and sends $AR \cap BC$ to the center of τ.

After that the line AR is the median so by Ceva theorem $\frac{AE}{EB} = \frac{DA}{CD}$ hence $ED \parallel BC$. As BC is a diameter of τ, $\sphericalangle PBC = 90^\circ = \sphericalangle BPE$ and $\sphericalangle BLE = 90^\circ$. Therefore, the quadrilateral $PBEL$ is cyclic. Analogously we can prove that $CQKD$ is cyclic. Let $T = PL \cap QK$, we have $\sphericalangle LTK = 180^\circ - (\sphericalangle TPQ + \sphericalangle PQT) = 180^\circ - (\sphericalangle LBA + \sphericalangle ACK) = \sphericalangle LAK$ which implies that T lies on τ. □



Problem 4. Given a $\triangle ABC$ with $\sphericalangle BAC = 90^\circ$ and two perpendicular lines x and y passing through the vertex A. For an arbitrary point X lying on the line x define y_b and y_c as the reflections of the line y about the line XB and XC respectively. Let Y be the common point of the lines y_b and y_c. Find the locus of the points Y.

Solution:

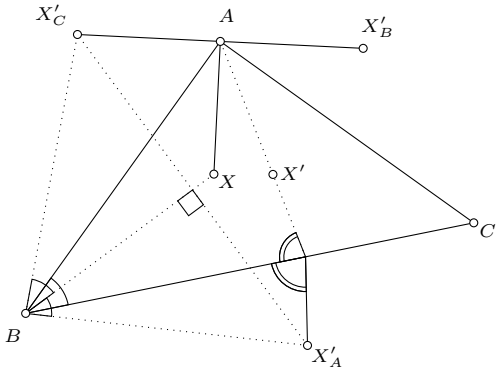
Consider the point X' isogonally conjugated to X with respect to $\triangle ABC$. Denote by $X'_A,\ X'_B$ and X'_C reflections of the point X about the lines $BC,\ AC$ and AB respectively. Perpendicularity of x and y implies that X'_B and X'_C lie on y.

Easy angle chasing gives

$$\sphericalangle X'_A BX = \sphericalangle CBA = \sphericalangle X'_C BX$$

Since $BX'_A = BX' = BX'_C$, the line BX is the bisector of the segment $X'_C X'_A$. Analogously CX is the bisector of the segment $X'_B X'_A$.

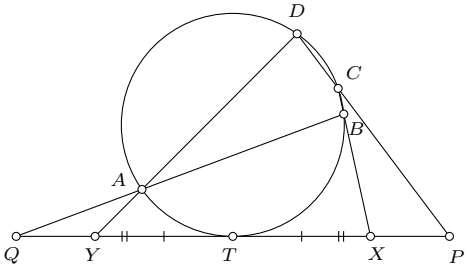
Therefore, X'_A lies on the lines y_b and y_c so it coincides with Y. It shows that Y lies on the reflection of the isogonal image of the line x about the line BC. The required locus is this line without the points such that y_b and y_c coincide, i.e. the common point of this line with BC and the reflection of A about BC.



Problem 5. Let $ABCD$ be the quadrilateral inscribed in the circle ω. Line l is tangent to the circle ω at the point T. The lines $AB,\ BC,\ CD$ and AD intersect the line l at the points $Q,\ X,\ P$ and Y respectively. Show that if $TY = TX$ then $TP = TQ$.

Solution:

Lets use the Desargues' Involution Theorem for the quadrilateral $ABCD$ and the conic ω to find the involution ϕ on the line l. The point T is constant with respect to ϕ and $X \leftrightarrow Y$. Since involution is determined by two reciprocal pairs of points, ϕ is the symmetry in point T. As we know $P \leftrightarrow Q$ so our proof is finished.



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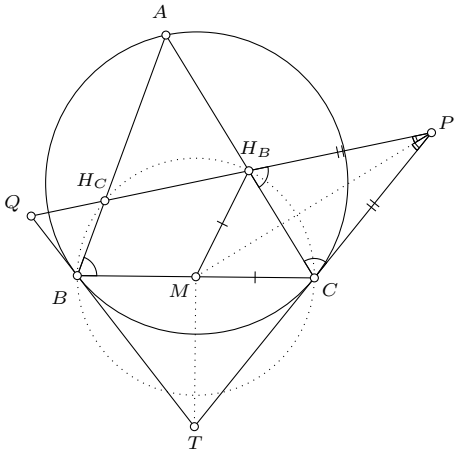
Problem 6. Let H_B and H_C be the foots of the altitudes from the vertexes B and C in $\triangle ABC$ respectively. Denote by Γ circumcircle of $\triangle ABC$. Prove that circumcircle of triangle bounded by the lines tangent to Γ at the points B

and C and the line H_BH_C is tangent to Γ.

Solution:



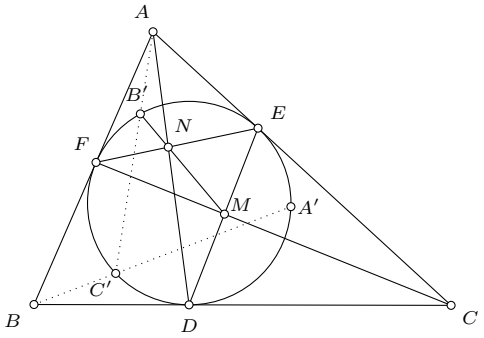
Let two given tangents intersect at the point T. Denote by M the midpoint of segment BC and let $P := TC \cap H_BH_C$ and $Q := TB \cap H_BH_C$. Using well-known mixtilinear circle property it is enough to prove that M is the incenter of $\triangle TPQ$.

Obviously, point M lies on the internal bisector of $\sphericalangle BTC$. On the other hand, $MC = MH_B$ and $\sphericalangle PH_BC = \sphericalangle ABC = \sphericalangle PCH_B$ so quadrilateral H_BMCP is a kite. It implies that MP is the inner bisector of $\sphericalangle H_BPC$ and ends our proof.

Problem 7. A scalene $\triangle ABC$ and its incircle ω are given. Using only a ruler and drawing at most eight lines, rays or segments, construct points A', B' and C' on ω such that the rays $B'C'$, $C'A'$ and $A'B'$ pass through the points A, B and C respectively.

Solution:

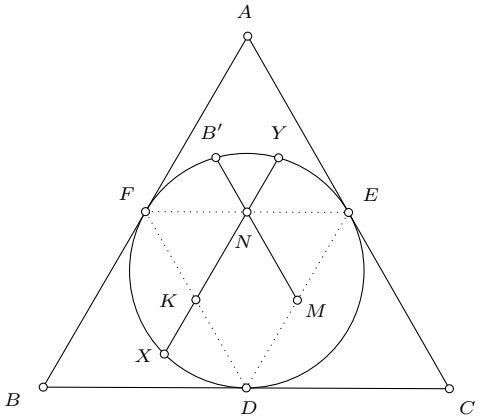
Let $\triangle DEF$ be the intouch triangle. Firstly, we will show the construction:



1.-2. Draw lines EF, AD and mark their intersection point N.
3.-4. Draw lines ED, CF and mark their intersection point M.
5. Draw line MN and mark intersection point of this line and ω - B'.
6. Draw line AB' and mark intersection point of this line and ω - C'.
7. Draw line BC' and mart intersection point of this line and ω - A'

Solution:

We will prove that this construction works. Consider projective transformation preserving circle ω end sending $AD \cap BE \cap CF$ into the center of ω.



Denote by K the midpoint of the segment DF. Points X and Y lies on the circle ω such that the points X, K, N and Y lie on the one line in this order.

By symmetry M and N are the midpoints of the segments DE and EF respectively. As $B'Y \parallel EF$ we have

$$-1 = Y(B', N; E, F) = (B', X; E, F)_\omega$$

so quadrilateral $B'FXE$ is harmonic. Hence, the points A, B' and X are collinear so $X = C'$.

Now it is easy to see that points B', A' and C are collinear.

Problem 8. Let ω be the incircle of $\triangle ABC$. The circle ω touches sides BC and AB at the points D and F respectively. Denote by D' the reflection of the point D with respect to the point C. The circle passing through the points B and C is tangent to ω at the point T. Let J be the A-excenter of $\triangle ABC$. Prove that the points T, F, B, J and D' lie on the one circle.

Solution:

First of all, denote by E the tangency point of the segment AC and ω. Moreover, let M and N be the midpoints of the segments DE and DF respectively.

The line TD is the bisector of $\sphericalangle BTC$ by homothety. Using symmedian properties we get $\sphericalangle FTN = \sphericalangle BTD = \sphericalangle DTC$. Furthermore, $\sphericalangle TDC = \sphericalangle TFD$ so

$$\triangle TFN \cup D \sim \triangle TDC \cup D'.$$

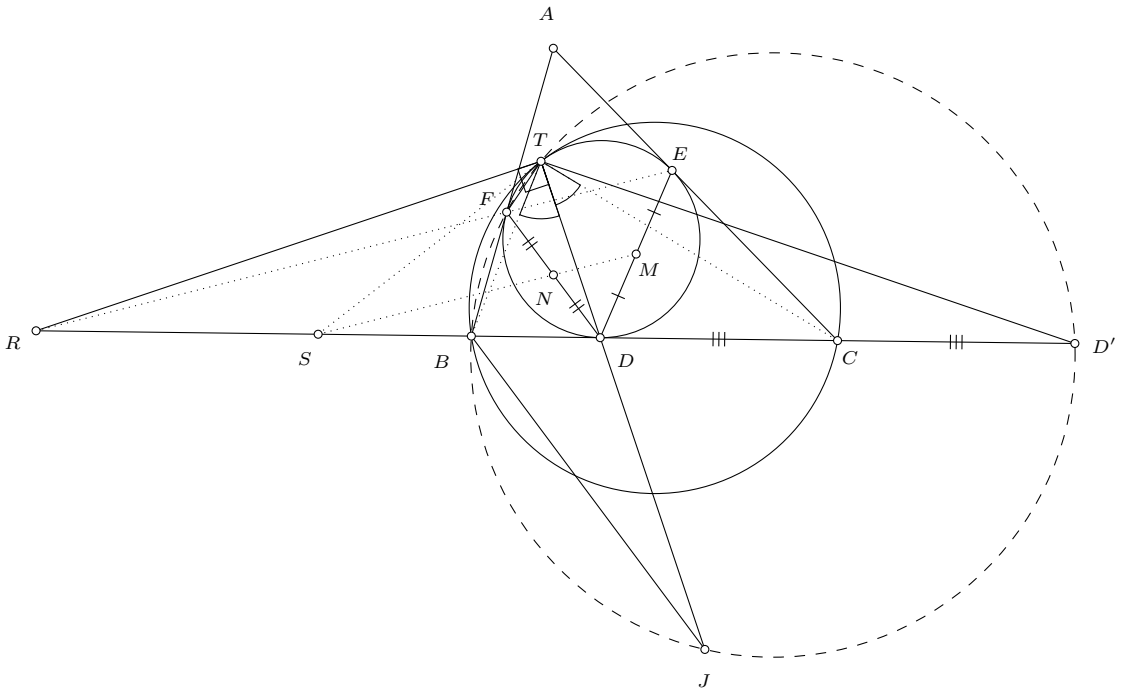
Hence, $\sphericalangle TFA = \sphericalangle TDF = \sphericalangle TD'D$ so the quadrilateral $BD'TF$ is cyclic.

Now we will prove collinearity of the points $T,\ D$ and J. Due to **MBL 2017 QQ Problem 7** the line MN is the polar line of the point J with respect circle ω. Choose the point S on the line BC such that the line ST is tangent to ω. Since the lines ST and SD are the polar lines of the points T and D respectively with respect circle ω, it is enough to prove collinearity of the points $M,\ N$ and S.

Denote by R and S' the intersection points of the lines EF and MN with the line BC respectively. Due to homothety with center D, the point S' is the midpoint of the segment RD. Notice that, $(R, D; B, C) = -1$ and $\sphericalangle BTD = \sphericalangle DTC$ so $\sphericalangle RTD = 90°$. Hence $S'T = S'D$.

On the other hand, $\sphericalangle STD = \sphericalangle SDT$ so $S = S'$. In ends a prove of collinearity of the point $T,\ D$ and J.

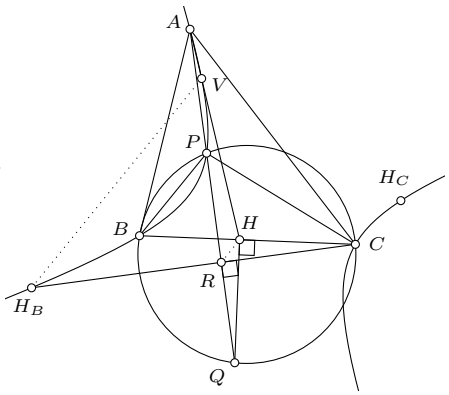
Finally, notice $FD \| BJ$ so $\sphericalangle TFA = \sphericalangle TDF = \sphericalangle TJB$ so the quadrilateral $BJTF$ is cyclic. It ends a proof of our problem.



Problem 9. Let P be any point inside $\triangle ABC$. Denote by H_C and H_B the orthocenters of $\triangle PAB$ and $\triangle PAC$ respectively. The line PA cuts circumcircle of $\triangle BPC$ again at the point Q. The point H is the projection of Q on the line BC. Prove that $AH \perp H_B H_C$.

Solution:
Let R be the projection of the point C onto the line AP. Note that $R \in CH_B$. Let V be the orthocenter of $\triangle AH_BH_C$ and denote by $H' := AV \cap BC$.

From Reim's Theorem for circumcircles of the quadrilaterals $BCPQ$ and $CHQR$ cut by the lines BH and PR, we get $BP \parallel HR$.



On the other hand, if $\mathcal{H}$ is the rectangular circumhyperbola of $\triangle ABC$ passing through the point P, then the points $K, L \in \mathcal{H}$ on account of rectangularity, which in turn gives $V \in \mathcal{H}$. Since $VH_B \perp AH_C \perp BP$, we have $BP \parallel H_BV$. Pascal's Theorem for AVH_BCBP gives collinearity of the points $R = AP \cap H_BC,\ H' = BC \cap AV$ and $X_\infty := H_BV \cap BP$. Hence, $H'R \parallel BP$ so $H' = H$. It implies that $AH \perp H_BH_C$.

□

Dukejukem - AoPS's user solution

Number Theory

Younger division

Problem 1. Let $n > 1$ be a positive integer. Tuple $(a_1, a_2, \ldots, a_n)$ of integer numbers greater than 1 is *good* if and only if

$$a_i \Big| \left(\frac{a_1 a_2 \cdots a_n}{a_i} - 1 \right)$$

for $i = 1, 2, \ldots, n$. Prove that there are only finitely many good n-tuples.

Solution:

As for all, assume in touple $(a_1, a_2, \ldots a_n)$ inequalities $a_1 \leq a_2 \leq \cdots \leq a_n$ occur. Suppose we have infinitely many such tuples. Since $i = 1, 2, \ldots, n$ $a_i \Big| \left(\frac{a_1 a_2 \cdots a_n}{a_i} - 1 \right)$ so product

$$\prod_{i=1}^{n} \frac{a_1 a_2 \cdots a_{i-1} a_{i+1} \cdots a_n - 1}{a_i}$$

is an integer number. It follows that the number

$$\frac{1}{a_1} + \ldots + \frac{1}{a_n} - \frac{1}{a_1 a_2 \ldots a_n}$$

is integer also.

On the other hand, $\frac{1}{a_1} + \ldots + \frac{1}{a_n} - \frac{1}{a_1 a_2 \ldots a_n} < n$ so minimal number - a_1 is bounded. Hence, there is exist set of tuples F_{a_1} for which a_1 is fixed and $|F_{a_1}| = \infty$. Analogously we can construct set $F_{a_2} \subseteq F_{a_1}$ such that all tuples in F_{a_2} has a_2 fixed and $|F_{a_2}| = \infty$ and so on. Finally, we construct family of sets $F_{a_1}, F_{a_2}, \ldots F_{a_n}$ such that

$$F_{a_n} \subseteq F_{a_{n-1}} \subseteq \cdots \subseteq F_{a_1}$$

and $|F_{a_n}| = \infty$. But F_{a_n} contains only one tuple contradiction.

Problem 2. Paweł and Ania are playing the following game. First Ania chooses a positive integer x less than 20170. Then Paweł asks her questions about greatest common divisor of numbers $x + n$ and m with n and m being positive integers less than 20170 of his choice. Prove that Paweł can determine this number after 6 such questions.

Solution:

Let's note that for each prime number p Paweł can find out residue of x modulo p in $p - 1$ moves. He simply plugs in $n = 1, 2, \ldots, p - 1$ and $m = p$ in his questions. Then if still has some time he can check out the the second power of p using exactly the same method - supposing that he knows that $p \mid x + k$, he checks $x + k + p, x + k + 2p, \ldots, x + k + p^2$ and can proceed so with any other power of p. Now since he has only 6 questions he can find out the residue of x modulo $2^6, 3^3, 5^1, 7^1$.

However, let's see that he can combine all these results using Chinese Remainder Theorem, in his every question n won't exceed m, which in turn won't exceed the maximum of $2^4 \cdot 3^2 \cdot 5 \cdot 7 = 5040 < 20170$ (in the third question, which is the last question about prime number 5) and $2^6 \cdot 3^3 \cdot 7 = 12096 < 20170$ (in the last question). At the same time, as a result he will get to know the remainder of x modulo $2^6 \cdot 3^3 \cdot 5 \cdot 7 = 60480 > 20170$, hence indeed using this strategy will determine Ania's number in 6 questions.

Problem 3. Let p be a prime number and f an integer polynomial of degree d such that $f(0) = 0, f(1) = 1$ and $f(n)$ is congruent to 0 or 1 modulo p for every integer n. Prove that $d \geqslant p - 1$.

Solution:

$$1^k + 2^k + \ldots + (p-1)^k \equiv 0 \pmod{p}$$

for all $k < p - 1$. Assuming that $\deg P < p - 1$ this yields

$$f(0) + f(1) + \ldots + f(p-1) \equiv 0 \pmod{p}.$$

Since $f(0) = 0, f(1) = 1$ and $f(k) \equiv 0$ or $1 \pmod{p}$, this is a contradiction.

Based on zhero's solution on AoPS

Problem 4. Show that the congruence $x^8 \equiv 16 \pmod{p}$ has a solution for every prime p.

Solution:

Note that, the case $\left(\frac{2}{p}\right) = 1$ is trivial, as well as the case $p = 2$. So let's assume that $\left(\frac{2}{p}\right) = -1$. It is satisfied if and only if $p \equiv 3 \vee 5 \pmod 8$. So we know that, $8 \nmid p - 1$, hence the classes of 4-th powers and 8-th powers are equal, i.e. function $f(x) = x^2$ is a bijection of the set $\{x \in \mathbb{F}_i^* | x \equiv y^4 \pmod p \text{ has a solution}\}$. Therefore 16 is an 8-th power modulo p.

Problem 5. Determine all functions f defined on the natural numbers that take values among the natural numbers for which

$$(f(n))^p \equiv n \mod f(p)$$

for all $n \in \mathbf{N}$ and all prime numbers p.

Solution:

Firstly, see that $f(p) \in \{1, p\}$ for every prime p. Indeed, $0 \equiv (f(p))^p \equiv p \pmod{f(p)}$ so $f(p) \mid p$. Now let's assume that there exist primes $q > r$ such that $f(q) = q$ and $f(r) = 1$. Then we obtain $(f(r))^q \equiv 1 \equiv r \pmod q$. But this is a contradiction since $1 < r < q$. Let's consider cases:

(i) (There exists a primes $p, q > 2$ such that $f(p) = p$ and $f(q) = 1$.)
Then of course $q > p$ and for every prime $q_0 > q$ $f(q_0) = 1$. Let's take a prime r greater than q that is not congruent to 1 modulo p (such a prime exists according to Dirichlet Theorem). We have:

$$1 \equiv (f(r))^p \equiv r \pmod p \Rightarrow \text{Contradiction.}$$

(ii) (There exists a prime $q > 2$ such that $f(q) = 1$ and $f(2) = 2$)
See that in this case every function that preserves parity satisfies the conditions and only such function do.

(iii) (For every prime p, $f(p) = 1$)
See that all functions that have the aformentioned property satisfy the conditions.

(iv) (For every prime p, $f(p) = p$)
Then easly we obtain that $f(n) = n$.

Problem 6. Let k be a positive integer. Show that there exist infinitely many pairs of integers (a, b) with $|a|, |b| > 1$ such that: $ab + b + a | a^2 + b^2 + k$.

Solution:

Let's see that in case of $k = 1$ any pair $(n^2, (n+1)^2)$ satisfies this equation:

$$a^2 + b^2 + 1 = n^4 + (n+1)^4 + 1 = 2n^4 + 4n^3 + 6n^2 + 4n + 2 = 2(n^4 + 2n^3 + 3n^2 + 2n + 1) =$$

$$2(n^2(n+1)^2 + n^2 + (n+1)^2) = 2(ab + a + b)$$

From now on we may therefore assume that $k > 1$. Note that $(1, 2k + 2)$ is always a solution

$$a^2 + b^2 + k = 1 + 4(k+1)^2 + k = 4k^2 + 9k + 5 = (k+1)(4k+5) = (k+1)(ab + a + b)$$

Suppose now we are given a solution a, b and let's assume without loss of generality that $a \leqslant b$. Then there exists a positive integer l such that

$$a^2 + b^2 + k = l(ab + a + b) \quad a^2 - (lb + 1)a + (b^2 - lb + k) = 0$$

We know that this quadratic equation has at least one integer root a. However, from Viete formulas we know tht $lb + 1 - a$ is its root as well. Furthermore, if $l > 1$, then $lb + 1 - a \geqslant 2b + 1 - a > a$, hence we obtain another solution with higher value of $a^2 + b^2$. Applying repeatedly this procedure we may therefore obtain infinitely many solution from just one solution (which we already have) for a given k provided $l > 1$. But for our solution $(1, 2k + 2)$ $l = k + 1 > 1$, hence it will be so for all other solution generated by it.

Problem 7. Given $p_1, p_2, ...$ be a sequence of integer and $p_1 = 2$. For positive integer n, p_{n+1} is the least prime factor of $np_1^{1!}p_2^{2!}...p_n^{n!} + 1$. Prove that all primes appear in the sequence.

Solution:

Suppose there exists smallest (odd) prime number p that never appears in the sequence. Let $m = \max\{C, p-1\}+1$ where C is the smallest positive integer such that all primes less than p appear in $p_1, p_2, ..., p_C$. We get that $a_h = 1 + h\prod_{i=1}^{h} p_i^{i!}$ never be a multiple of p for all $h > m$ By FLT, it's equivalent to $1 + hT$ never be a multiple of p where $T = \prod_{i=1}^{m} p_i^{i!}$. Note that $p \nmid T$, we get a contradiction, done.

Based on ThE-dArK-lOrD's solution from AoPS

Problem 8. Let p be an odd prime and r an odd natural number.Show that $pr + 1$ does not divide $p^p - 1$.

Solution:

Let $d \in \mathbb{N}$ be a divisor of $p^p - 1$ with $d \equiv 1 \pmod{p}$. Let S be the set of all prime divisors of d that also divide $p - 1$. We will show that S is empty. Suppose otherwise, and note that by LTE, we have $v_q(p^p - 1) = v_q(p - 1)$ for all $q \in S$. Consequently,

$$1 < \prod_{q\in S} q^{v_q(d)} \leq \prod_{q\in S} q^{v_q(p^p-1)} \leq \prod_{\substack{q|p-1 \\ q \text{ prime}}} q^{v_q(p-1)} = p - 1.$$

Now, let T be the set of all prime divisors of d that do not divide $p - 1$. Then for any $q \in T$, we have $q \mid p^p - 1 \implies \mathrm{ord}_q(p) \mid p \implies \mathrm{ord}_q(p) \in \{1, p\}$. However, $\mathrm{ord}_q(p) \neq 1$ by assumption. Therefore, by Fermat's Little Theorem, we obtain $p = \mathrm{ord}_q(p) \mid q - 1 \implies q \equiv 1 \pmod{p}$. It follows that

$$1 \equiv d = \left(\prod_{q\in S} q^{v_q(d)}\right)\left(\prod_{q\in T} q^{v_q(d)}\right) \equiv \prod_{q\in S} q^{v_q(d)} \not\equiv 1 \pmod{p},$$

a contradiction. Consequently, S is empty. It follows that

$$d \mid \frac{p^p - 1}{p - 1} = 1 + p + \cdots + p^{p-1},$$

which is odd. Consequently, d is odd as well. Therefore, if we write $d = pr + 1$, it follows that r cannot be odd. □

Based on Dukejukem's solution on AoPS

Problem 9. Let be P be a polynomial with integer coefficients such that for every natural n we have $P(n) > n$. Moreover, if x_k is a sequence that: $x_1 = 1, x_{i+1} = P(x_i)$ then for every natural number N one of x_i is divisible by N. Prove that $P(x) = x + 1$.

Solution:

We will use fact that $a - b|P(a) - P(b)$ for $a, b \in \mathbb{Z}_+$ and $a \neq b$.

First of all, there exists a number B such that for all integer $n > B$ we have $x_n > x_i$ for all integer $0 < i < n$. Choose $n > B + 1$, then there exists $m \in \mathbb{Z}_+$ such that $x_n - x_{n-1}|x_m$. Consider cases:

(a) $(m \geq n)$ We see that $x_n - x_{n-1}|P(x_n) - P(x_{n+1}) = x_{n+1} - x_n| \ldots |x_m - x_{m-1}$. Hence:

$$x_m \equiv x_{m-1} \equiv \ldots \equiv x_{n-1} \pmod{x_n - x_{n-1}}$$

Since $x_n - x_{n-1}|x_m$ we have $x_n - x_{n-1}|x_{n-1}$ so $x_n \leq 2x_{n-1}$.

(b) $(m < n)$ We see that $x_n - x_{n-1}|x_m$ implies $x_n \leq x_{n-1} + x_m < 2x_{n-1}$ as $n - 1 > B$.

Since x_i for $i > B$ is strictly increasing we proved that for arbitrary large n we have $P(n) \leq 2n$ so P has degree at most 1. Polynomial P can not be constant so $P = ax + b$ for some $a \in \mathbb{Z}_+$ and $b \in \mathbb{Z}$.

We have $2 = 2 \cdot x_1 \geq x_2 = P(1) > 1$ so $a + b = P(1) = 2$. Moreover, $4 = 2 \cdot x_2 \geq x_3 = P(2) > 2$ so $2a + b = 3$ or $2a + b = 4$. Therefore $P(x) = x + 1$ or $P(x) = 2x$. In second case $x_n = 2^{n-1}$ and x_i sequence doesn't satisfy the conditions.

Older division

Problem 1. Find all $f(x) \in \mathbb{Z}[x]$ such that $f(p)|(2^p - 2)$ for all prime number p.

Solution:

If $x^k \mid f(x)$, note that k is either one or zero as not all primes are Wieferich primes. If f is constant it's either ± 2 or ± 1 so assume it's non-constant and let $f(0) \neq 0$.By Schur Theorem f has infintely many prime divisors so pick one say q larger than $f(0)$ and say $q \mid f(r)$. Pick such prime p that $(p-1, q-1) = 0$ and $p \equiv r \pmod{q}$ (we can find such a prime according to Dirichet Theorem) which provides contradiction as $q \mid 2^{p-1} - 1$ but $(q-1, p-1) = 1$.Now say $f(0) = 0$ and let $f(x) = xg(x) \implies g(p) \mid 2^p - 2$ by previous g is in the set $\{\pm 2, \pm 1\}$ .g isn't ± 2 because of $p = 2$ which leaves: $f(X) \in \{\pm X, \pm 2, \pm 1\}$

Based on nikolapavlovic's solution on AoPS

Problem 2. Let P be a polynomial with integer coefficients such that $P(0) = 0$ and

$$\operatorname{nwd}(P(0), P(1), P(2), \ldots) = 1.$$

Show there are infinitely many n such that

$$\operatorname{nwd}(P(n) - P(0), P(n+1) - P(1), P(n+2) - P(2), \ldots) = n.$$

Solution:

Let $P(x) = x^k Q(x)$, where $Q(0) \neq 0$. We claim that if $n = p^k$ for any prime p not dividing $kQ(1) + Q'(1)$, then n will satisfy the conditions of the problem.

It is obvious that $p^k | P(p^k+i) - P(i)$ for all i, whence $p^k | \operatorname{nwd}(P(p^k+i) - P(0), P(p^k+1) - P(1), P(p^k+2) - P(2), \ldots)$. We now claim that $p^{k+1} \nmid P(p^k+1) - P(1)$. If we let $Q(x) = \sum_{i=0}^{m} c_i x^m$, then

$$\begin{aligned}
P(p^k+1) - P(1) &= (p^k+1)^k Q(p^k+1) - Q(1) \\
&= \left(\sum_{i=0}^{k} \binom{k}{i} (p^k)^i \right) Q(p^k+1) - Q(1) \\
&\equiv (kp^k + 1) Q(p^k+1) - Q(1) \\
&= kp^k Q(p^k+1) + (Q(p^k+1) - Q(1)) \\
&= kp^k Q(p^k+1) + \sum_{i=0}^{m} c_i ((p^k+1)^i - 1^m) \\
&= kp^k Q(p^k+1) + \sum_{i=0}^{m} c_i \left(-1 + \sum_{j=0}^{i} \binom{i}{j} (p^k)^j \right) \\
&\equiv kp^k Q(p^k+1) + \sum_{i=0}^{m} i c_i p^k \\
&= p^k (kQ(p^k+1) + Q'(1)) \pmod{p^{k+1}}.
\end{aligned}$$

Hence,

$$\frac{P(p^k+1) - P(1)}{p^k} = kQ(p^k+1) + Q'(1) \equiv kQ(1) + Q'(1) \not\equiv 0 \pmod{p},$$

so p^k fully divides $\operatorname{nwd}(P(p^k+i) - P(0), P(p^k+1) - P(1), P(p^k+2) - P(2), \ldots)$.

It therefore suffices to show that we cannot find some $q \neq p$ dividing each of $P(p^k) - P(0), P(p^k+1) - P(1), P(p^k+2) - P(2), \ldots$. Suppose for the sake of contradiction that for all positive integers i, $q | P(p^k+i) - P(i)$. Observing also that $q | P(q+i) - P(i)$, we find that $P(i + ap^k + bq) \equiv P(i) \pmod{q}$ for all integers a and b. Since $\operatorname{nwd}(p^k, q) = 1$, we can pick a and b so that $ap^k + bq = 1$, yielding $q | P(i+1) - P(i)$ for all i. But $q | P(0) = 0$, so $q | P(i)$ for all nonnegative i, which contradicts the assumption.

Based on Zhero's solution on AoPS

Problem 3. Show that there exist only finitely many positive integers n such that

$$\varphi(n) < n^{\frac{1}{2017}}.$$

Solution:

First of all, lets make some observations $\varphi(n) = n\prod_{p|n}\left(1-\frac{1}{p}\right) < n^{\frac{1}{2017}}$ so $\prod(1-\frac{1}{p}) < \prod p^{\frac{1}{2017}-1}$ and $\prod \frac{p-1}{p^{\frac{1}{2017}}} < 1$.

Since linear function $p-1$ increase faster that exponential $p^{\frac{1}{2017}}$, we have inequality $p-1 < p^{\frac{1}{2017}}$ only for finitely many prime numbers p. Denote by

$$\mathcal{A} = \{p : p \text{ is prime and } p-1 < p^{\frac{1}{2017}}\}, \qquad A = \prod_{p\in A}\frac{p-1}{p^{\frac{1}{2017}}}.$$

Similarly there are finitely primes, such that $\frac{A(p-1)}{p^{\frac{1}{2017}}} < 1$. Therefore, there are finitely number of sets $\mathcal{S}$ of prime numbers, such that $\prod_{p\in\mathcal{S}}\left(1-\frac{1}{p}\right) < \prod_{p\in S} p^{\frac{1}{2017}}$.

For each set $\mathcal{S}$ there are finitely many natural of sequences $(a_1, a_2, \ldots, a_{|\mathcal{S}|})$, such that $n = \prod_{p\in\mathcal{S}} p^{a_p}$ and $\frac{\varphi(n)}{n^{\frac{1}{2017}}} = \prod_{p\in\mathcal{S}}\left(1-\frac{1}{p}\right)p^{a_p(1-\frac{1}{2017})} < 1$ (We can fix the $\sum_{i=1}^{|\mathcal{S}|} a_i$ and estimate product of primes by powers of 2).

It proved that there are finitely n, such that $\varphi(n) < n^{\frac{1}{2017}}$.

Problem 4. Let $n > 1$ be a given integer. Prove that infinitely many terms of the sequence $(a_k)_{k\geq 1}$, defined by

$$a_k = \left\lfloor \frac{n^k}{k} \right\rfloor,$$

are odd. (For a real number x, $\lfloor x \rfloor$ denotes the largest integer not exceeding x.)

Solution:

Take $s = (q-1)^2$ for some prime $q \geqslant 3$ and $(q, n) = 1$.

By Zsigmondy there is a prime divisor p of $n^q - 1$ that does not divide $n-1$. So, q is the order of n modulo p.

Select $k = n^s p$.

Then $n^s p - s \equiv n^s - s = n^{(q-1)^2} - (q-1)^2 \equiv 1 - 1 \equiv 0 (mod q)$. So, $n^{n^s p} \equiv n^s (mod p)$.

This implies $\left\lfloor \frac{n^k}{k} \right\rfloor = \left\lfloor \frac{n^{n^s p - s}}{p} \right\rfloor = \frac{n^{n^s p - s} - 1}{p}$ which is clearly odd.

Based on i_ganev's solution on AoPS

Problem 5. Define the sequence $a_1, a_2, a_3, \ldots$ by $a_1 = 1$ and, for $n > 1$,

$$a_n = a_{\lfloor n/2 \rfloor} + a_{\lfloor n/3 \rfloor} + \ldots + a_{\lfloor n/n \rfloor} + 1.$$

Prove that there are infinitely many n such that $a_n \equiv n \pmod{2^{2017}}$.

Solution:

Let $M = 2017$.

Let $d_n = a_n - a_{n-1}$ for $n > 1$ and $d_1 = 1$. Note that for $n > 1$,

$$\begin{aligned} d_n &= a_n - a_{n-1} \\ &= (a_{\lfloor n/2 \rfloor} - a_{\lfloor (n-1)/2 \rfloor}) + (a_{\lfloor n/3 \rfloor} - a_{\lfloor (n-1)/3 \rfloor}) + \ldots + (a_{\lfloor n/(n-1) \rfloor} - a_{\lfloor (n-1)/(n-1) \rfloor}) + a_{\lfloor n/n \rfloor}. \end{aligned}$$

For all integers $k \nmid n$, $a_{\lfloor n/k \rfloor} = a_{\lfloor (n-1)/k \rfloor}$ and thus cancel out in the sum. For all $k|n$,

$$a_{\lfloor n/k \rfloor} - a_{\lfloor (n-1)/k \rfloor} = a_{n/k} - a_{n/k-1} = d_{n/k}.$$

Thus,

$$d_n = \sum_{k|n, k<n} d_k.$$

Lemma 1. Suppose that $p^k | m$ for a prime p and positive integer k. Then, $2^{k-1} | d_m$. Proof. The proof will be by induction on m. For $m = 2$, the statement follows from the fact that the only prime factor is 2 and thus,

$2^{1-1} = 1|d_2$. Suppose the statement holds for all $m < n$. Now, let $m = n$. If $k = 1$, then the statement follows trivially. Suppose that $k \geq 2$. Note that

$$\begin{aligned} d_n &= \sum_{i|n, i<n} d_i \\ &= d_{n/p} + \sum_{i|(n/p), i<n/p} d_i + \sum_{i|(n/p^k), i<n/p^k} d_{i\cdot p^k} \\ &= d_{n/p} + d_{n/p} + \sum_{i|(n/p^k), i<n/p^k} d_{i\cdot p^k} \\ &= 2d_{n/p} + \sum_{i|(n/p^k), i<n/p^k} d_{i\cdot p^k}. \end{aligned}$$

Note that every value in the summation (if any exist)

$$\sum_{i|(n/p^k), i<n/p^k} d_{i\cdot p^k}$$

is divisible by 2^{k-1} by the inductive assumption. Thus,

$$d_n \equiv 2d_{n/p} \pmod{2^{k-1}}.$$

Note that since $p^{k-1}|(n/p)$, by the inductive assumption, $2^{k-2}|d_{n/p}$. Thus, $2^{k-1}|2d_{n/p}$ and therefore $2^{k-1}|d_n$, which completes the induction.■

Let $p_1, p_2, ..., p_{2^M}$ be 2^M distinct odd primes greater than 2^M. By the chinese remainder theorem, there exists infinitely many integers n greater than 2^M such that

$$n \equiv i - 1 \pmod{p_i^{M+1}}$$

for all $1 \leq i \leq 2^M$. Note that by Lemma 1, since $p_i^{M+1}|(n - i + 1)$, $d_{n-i+1} \equiv 0 \pmod{2^M}$ for all $1 \leq i \leq 2^M$. Thus,

$$a_n \equiv a_{n-1} \equiv ... \equiv a_{n-2^M+1} \equiv c \pmod{2^M}$$

where c is some integer. Note that among the integers $n, n-1, ..., n-2^M+1$, there exists exactly one that is congruent to c modulo 2^M. Let this integer be l. Then, $a_l \equiv c \equiv l \pmod{2^M}$. Thus, there exists infinitely many n such that $a_n \equiv n \pmod{2^M}$.

Note. The sequence d_n counts the number of sequences $1 = k_1 < k_2 < ... < k_t = n$ such that $k_i|k_{i+1}$ for all $1 \leq i \leq t-1$. Thus, the sequence a_n counts the number of sequences $1 = k_1 < k_2 < ... < k_t \leq n$ such that $k_i|k_{i+1}$ for all $1 \leq i \leq t-1$.

Based on HroK's solution on AoPS

Problem 6. Let the function $f : N^* \to N^*$ such that (1) $(f(m), f(n)) \leq (m, n)^{2014}, \forall m, n \in N^*$; (2) $n \leq f(n) \leq n+2014, \forall n \in N^*$ Show that: there exists the positive integers N such that $f(n) = n$, for each integer $n \geq N$.

Solution:

Lemma 1: For every prime p and positive integer n, we have that $p \mid f(n) \Rightarrow p \mid n$.

Proof:

Assume for the sake of contradiction that there exists an $n \in \mathbb{N}$ and a prime p such that $p|f(n)$ and $(n, p) = 1$. Because $(f(n), f(p^k)) \leq (n, p^k)^{2014} = 1$ we have that $(f(n), f(p^k)) = 1$ for any $k \in \mathbb{N}$ and so $p \nmid f(p^k)$ for any $k \in \mathbb{N}$. Let $A = \{f(p^k) : k \in \mathbb{Z}^+\}$ and let $B = \{q : \exists a \in A \text{ such that } q|a, q \text{ is prime}\}$.

Assume for the sake of contradiction that set B contains infinitely many elements. Note that $p \notin B$. Consider 2015 distinct primes from B and label them $q_1, q_2, \ldots, q_{2015}$. By CRT there exists an $m \in \mathbb{N}$ such that $m \equiv 1 \mod p$ and $m \equiv -i + 1 \mod q_i$ for all $i \in \{1, 2, \ldots, 2015\}$. But since $m \leq f(m) \leq m + 2014$ this means that there exists an $i \in \{1, 2, \ldots, 2015\}$ such that $q_i|f(m)$. Now take a $k \in \mathbb{Z}^+$ such that $q_i|f(p^k)$ (we know that such a k exists from the definition of q_i). We have that $q_i \leq (f(m), f(p^k)) \leq (m, p^k)^{2014} = 1$, contradiction. This shows that B has finitely many elements, so let $B = \{q_1, q_2, \ldots, q_r\}$ for some $r \in \mathbb{Z}^+$.

Now I claim that for any $s \in \mathbb{Z}^+$, there exist primes $x_1, x_2, \ldots, x_s \notin B$ such that there exists a $k \in \mathbb{N}$ such that $p^k + i \equiv 0 \mod x_i$ for all $i \in \{1, 2, \ldots, s\}$. I prove this claim by induction. For the base case of $s = 1$, note that by Kobayashi's Theorem since the set of prime factors of elements of the sequence p^k for $k = 1, 2, \ldots$ is finite, the set of prime factors of the sequence $p^k + 1$ for $k = 1, 2, \ldots$ is infinite which allows us to pick a suitable k

and x_1. For the inductive step, assume we have a k such that $p^k + i \equiv 0 \mod x_i$ for every $i \in \{1, 2, 3, \ldots, t\}$ for some positive integer $t < s$. Then Kobayashi's Theorem again on the sequence $p^{k+(x_1-1)(x_2-1)\ldots(x_t-1)j} + t + 1$ for $j = 1, 2, \ldots$ allows us to pick a suitable j and x_{t+1}.

Using this fact with $s = 2014$ allows us to pick a k such that every integer in the interval $[p^k, p^k + 2014]$ is divisible by some x_i which contradicts the fact that these x_i are not in B. Hence the lemma is proven.

Lemma 2: f is injective on the interval $[2014^{2014}, \infty)$

Proof:

Assume there exist distinct integers $a, b > 2014^{2014}$ such that $f(a) = f(b)$. Then $f(a) = (f(a), f(b)) \leq (a, b)^{2014} \leq 2014^{2014}$ (since $|a - b| \leqslant 2014$), contradiction.

Lemma 2 implies that if there exists an integer $m \geq 2014^{2014}$ such that $f(m+i) = m+i$ for all $i \in \{1, 2, \ldots, 2014\}$ then $f(n) = n$ for all $n \in [2014^{2014}, m]$.

If a prime $p > 2014$ divides $f(n)$ for some $n \in \mathbb{N}$ then by Lemma 1 we have that $p|f(n) - n \leq 2014 \implies f(n) = n$. Therefore $f(n) \neq n$ only if all prime factors of $f(n)$ are less than 2014. Now it is a standard result that the gaps between consecutive 2014-smooth numbers (numbers with no prime factor greater than 2014) grow arbitrarily large (otherwise the sum of their reciprocals would diverge) and coupling this with the observation in the last paragraph we obtain the desired result. In fact, we have proven that $N = 2014^{2014}$ is suitable.ill

Based on Wostenholme's solution on AoPS

Problem 7. For a prime p, a subset S of residues modulo p is called a sum-free multiplicative subgroup of $\mathbb{F}_p$ if

(a) there is a nonzero residue α modulo p such that $S = \{1, \alpha^1, \alpha^2, \ldots\}$ (all considered mod p), and

(b) there are no $a, b, c \in S$ (not necessarily distinct) such that $a + b \equiv c \pmod{p}$.

Prove that for every integer N, there is a prime p and a sum-free multiplicative subgroup S of $\mathbb{F}_p$ such that $|S| \geq N$.

Solution:

The key insight to this problem is to transfer it over the the complex numbers, where roots of unity behave a lot more nicely. Fix $q > \max(N, 3)$ and let S be the set of q^{th} roots of unity for a prime p. Now if S were not sum free modulo p, there must exist a, b such that:

$$1 + \alpha^a \equiv \alpha^b \pmod{p}$$

Exponentiation by q:

$$(1 + \alpha^a)^q \equiv 1 \pmod{p}$$

Now, I claim that $(1 + x^a)^q - 1$ and $x^q - 1$ are relatively prime polynomials over $\mathbb{Z}[x]$. This is clear as we have $1 + \omega^a$ is a q^{th} root where ω is a q^{th} root of unity, but this is impossible for $q > 3$. Thus there exist $m(x), n(x) \in \mathbb{Z}[x]$ and an integer c_a such that:

$$m(x)((1 + x^a)^q - 1) + n(x)(x^q - 1) = c_a$$

Now if $p > c_a$, we get p cannot simultaneously divide $(1 + x^a)^q - 1$ and $x^q - 1$. Thus if we set $p \equiv 1 \pmod{q}$ with $p > c_0, c_1, ..., c_{q-1}$, we get S must be sum free as desired.

Based on dinoboy's solution on AoPS

Problem 8. Let $n \geq 2$ be an integer. Prove that if $k^2 + k + n$ is prime for all integers k such that $0 \leq k \leq \sqrt{\frac{n}{3}}$, then $k^2 + k + n$ is prime for all integers k such that $0 \leq k \leq n - 2$.

Solution:

Our strategy is to suppose some nonempty subset of the $k^2 + k + p$ to be composite, while still assuming the $0 \leqslant k \leqslant 3$ cases are prime. Among the prime factors of these composite numbers, there must exist the least prime q. Our approach is to bound q from either side until it breaks. Firstly (noting that the $p = 2$ case is trivial),

$$q \leqslant \sqrt{k^2 + k + p} \leqslant \sqrt{(p-2)(p-1) + p} < p.$$

Therefore, the interval $0 \leqslant k \leqslant p - 2$ contains all the residue classes modulo q. We consider the equation $x^2 + x + p = 0$ over the field $\mathbb{F}_q$. We are assuming it has a root, by choice of q. It therefore splits into two linear factors, and hence has two roots α_1, α_2 wlog (moving back to $\mathbb{Z}$) satisfying $0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant q - 1$. Furthermore,

we see that $\alpha_1 + \alpha_2 \equiv -1(\mod q)$, which is easily seen to imply $\alpha_1 + \alpha_2 = q - 1$, in turn implying (a fact which will be crucial to our strategy) that

$$\alpha_1 \leqslant \frac{q-1}{2}.$$

First, we bound q below. By assumption, $\alpha_1 > \sqrt{\frac{p}{3}}$ (q, being less than p, cannot possibly divide any of the prime numbers $k^2 + k + p$, $k \leqslant p3$). Therefore $\sqrt{\frac{p}{3}} < \alpha_1 \leqslant \frac{q-1}{2}$, which implies

$$q > 2\sqrt{\frac{p}{3}} + 1.$$

Next, we construct a sequence of upper bounds for q^2 , (B_i), based upon the following reasoning: By the minimality of q, $q^2 \leqslant \alpha_1^2 + \alpha_1 + p \leqslant \frac{q^2-2q+1+2q}{4} + p = \frac{q^2+1}{4} + p$.

Thus, if $q^2 \leqslant B_j$, $q_2 \leqslant \frac{B_j+1}{4} + p = B_{j+1}$. This equation becomes $4(B_{j+1} - B_j) = 4p + 1 - 3B_j$, so the bounds will keep getting smaller until $B_k \leqslant 4p + 1$for some k. It follows immediately that

$$q \leqslant \sqrt{\frac{4p+1}{3}} < 2\sqrt{\frac{p}{3}} + 1.$$

Contradiction.

Based on Ilthigore's solution on AoPS

Problem 9. Let $a > 1$ be a positive integer and $f \in \mathbb{Z}[x]$ with positive leading coefficient. Let S be the set of integers n such that

$$n \mid a^{f(n)} - 1.$$

Prove that S has density 0; that is, prove that $\lim_{n\to\infty} \frac{|S\cap\{1,\ldots,n\}|}{n} = 0$.

Solution:

Instead of considering the condition $n|a^{f(n)} - 1$ we just need to check the condition $ord_n(a)|f(n)$. Putting $n = pk$, where p is a prime, we get $ord_{pk}(a)|f(pk)$. It can't hold without $ord_p(a)|f(k)$, because $ord_p(a)|\ p-1\ |f(pk) - f(k)$ and $ord_p(a)|f(pk)$, so indeed $ord_p(a)|f(k)$. Now we will prove that this condition doesn't hold for $deg\ f + 1$ consecutive numbers k and for large primes. Suppose that: $ord_p(a)|f(k),\ f(k+1), \ldots, f(k + deg\ f)$ Then:

$$ord_p(a)|\sum_{i=0}^{deg\ f} (-1)^i \binom{deg\ f}{i} f(k+i)$$

That sum is equal to $(deg\ f)! * a_{deg\ f}$. It can't be divisible by $ord_p(a)$ for large primes. So for big primes p, there are at least $\frac{p}{deg\ f+1}$ residues k $modp$ such that the condition $ord_p(pk)|f(pk)$ isn't true. Let $c = \frac{1}{deg\ f+1}$. Then for big primes p, there are at least cp resiudes mod p^2, for which the condition of problem doesn't hold. Let the set of these residues be A_p. According to Chinese Remainder Theorem, the density of numbers, which doesn't belong to any A_p, is less than:

$$\prod_{big\ primes} (1 - \frac{c}{p})$$

. Instead of proving that this product is divergent to 0, we can show that the following product is divergent to infinity:

$$\prod_{big\ primes} (1 + \frac{c}{p})$$

. This is equivalent to prove that:

$$\sum_{big\ primes} \frac{c}{p} = \infty$$

. But this is a well-known result of Euler, so that's it.

Jan Fornal's solution

Mathematical Matches

Younger Match

Algebra

Problem 1. Find all triples (f, g, h) of injective functions from the set of real numbers to itself satisfying

$$f(x + f(y)) = g(x) + h(y)$$
$$g(x + g(y)) = h(x) + f(y)$$
$$h(x + h(y)) = f(x) + g(y)$$

for all real numbers x and y. (We say a function F is injective if $F(a) \neq F(b)$ for any distinct real numbers a and b.)

Solution:

Lets take a look at the following expression

$$Q(x, y, z) = f(x + g(z) + f(y)) = g(x + g(z)) + h(y) = h(x) + h(y) + f(z)$$

We see that $Q(x, y, z) = Q(y, x, z)$ so $f(x+g(z)+f(y)) = f(y+g(z)+f(x))$ and by injectivity $x+f(y) = y+f(x)$ so $f(x) = x + k$ for some real number k. Analogously $g(x) = x + l$ and $h(x) = x + m$ for some real numbers l and m. After putting it into our equations we obtain that only solution is $f(x) = g(x) = h(x) = x + c$ for some real number c.

Problem 2. Find all surjective functions $f : (0, +\infty) \to (0; +\infty)$ satisfying

$$2xf(f(x)) = f(x)(x + f(f(x))).$$

Solution:

Let's notice that this equation is equivalent to

$$\frac{1}{x} + \frac{1}{f(f(x))} = \frac{2}{f(x)}$$

Since the function is surjective (and also injective as we can see from the above equation), there exists its inverse $g(x)$. Due to the above equation we obtain that $\ldots, \frac{1}{g(g(x))}, \frac{1}{g(x)}, \frac{1}{x}, \frac{1}{f(x)}, \frac{1}{f(f(x))}, \ldots$ is an infinite (in both ways) arithmetic progression in positive real numbers. Since it is infinite in both directions, it follows that this must be a constant arithmetic progression and hence $f(x) = x$ for all $x \in (0, +\infty)$.

Combinatorics

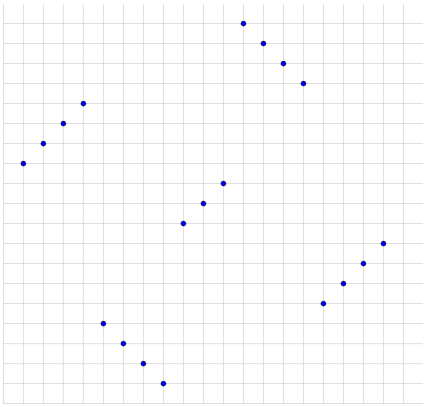
Problem 1. Let S be a set of 2017 points in the coordinate plane, no two of which have the same $x-$ or $y-$coordinate. For any two points $P, Q \in S$, consider the rectangle with one diagonal PQ and the sides parallel to the axes. Denote by W_{PQ} the number of points of S lying in the interior of this rectangle. Determine the maximum N such that, no matter how the points of S are distributed, there always exist points P, Q in S with $W_{PQ} \geq N$.
Solution:

Let's first notice that it suffices for every $k \in \mathbb{Z}_+$ to find $n(k)$, which is the largest number of points with distinct $x-$ and $y-$coordinates such that every rectangle spun by two of these points contains at most k other points.

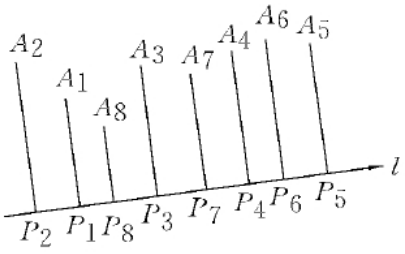
We shall prove that $n(k) = 5k + 4$. Suppose that we are given a set S such that every rectangle spun by two points from it contains at most k other points from S. Let A, B, C, D be points from S with the lowest $x-$, lowest $y-$, highest $x-$ and highest $y-$ coordinate, respectively. Furthermore, let L be the rectangle with sides parallel to $x-$ and $y-$axes such that points A, B, C, D lie on its border. Notice that rectangles with diagonals AB, BC, CD, DA and AC cover the whole rectangle L. At the same time, due to the definition of points A, B, C, D all points from S lie in the interior of L. Hence, we obtain that $|S| \leqslant 5k + 4$ (where 4 comes from

points A, B, C, D, since inside they are not inside any of the described rectangles) and thus $n(k) \leq 5k + 4$.

To see that $n(k)$ indeed equals $5k+4$, let's note that the construction for $k = 3$ presented on the right. In general, it yields a set S of $5k+4$ points such that every rectangle spun by two points of S contains at most k other points from S. Therefore, getting back to original problem it follows that no matter how we distribute points of S, where $|S| = 2017$, there will exist two points P, Q from S such that $W_{PQ} \geq \lfloor \frac{2017}{5} \rfloor = 403$ and 403 is the greatest number that satisfies this.



Problem 2. Assume that $A_1, A_2, \ldots, A_8$ are eight points taken arbitrarily on a plane. For a directed line l taken arbitrarily on the plane, assume that projections of $A_1, A_2, \ldots, A_8$ on the line are $P_1, P_2, \ldots, P_8$ respectively. If the eight projections are pairwise disjoint, they can be arranged as $P_{i_1}, P_{i_2}, \ldots, P_{i_8}$ according to the direction of line l. Thus we get one permutation for $1, 2, \ldots, 8$, namely, $i_1, i_2, \ldots, i_8$. In the figure, this permutation is $2, 1, 8, 3, 7, 4, 6, 5$. Assume that after these eight points are projected to every directed line on the plane, we get the number of different permutations as $N_8 = N(A_1, A_2, \ldots, A_8)$. Find the maximal value of N_8.



Solution:

First, let's notice that if we choose some line passing through the origin, then rotating it around the origin and looking on the corresponding orders of projections we will obtain all the possible orders. Every two permutations that "neighbour" in this way are bordered by a permutation with two points projected to one point - that is, they are bordered by a line which is perpendicular to one of $\binom{8}{2}$ segments between points.

At the same time, every such line is a border between two neighbouring permutations exactly twice (since for every undirected line we can choose its direction in two ways) and every permutation is bounded by two such permutations. Therefore, since there are at most $\binom{8}{2}$ such lines (with equality hapenning if no two segments between points of $A_1, A_2, \ldots, A_8$), there are at most $2\binom{8}{2} = 56$ such permutations, where this number is attained if and only if no two segments between two points from $A_1, A_2, \ldots, A_8$ are parallel.

Problem 3. Given a finite sequence of real numbers $a_1, a_2, \ldots, a_n$ $(*)$, we call a segment $a_k, \ldots, a_{k+l-1}$ of the sequence $(*)$ a *dragon* and a_k is a *head* of the dragon if the arithmetic mean of $a_k, \ldots, a_{k+l-1}$ is greater than 2017 (especially if a single item $a_m > 2017$, we still regard a_m as a dragon). Suppose that there is at least one dragon among the sequence $(*)$, show that the arithmetic mean of all those items of sequence $(*)$ that could be head of a certain dragon (that is, the arithmetic mean of all the heads, counted with multiplicities) is greater than 2017.
Solution:

Suppose $a_k, \ldots, a_{k+l-1}$ is a dragon, as well as $a_{k+1}, \ldots, a_{k+l-1}$, $a_{k+2}, \ldots, a_{k+l-1}$, $\ldots$, $a_m, \ldots, a_{k+l-1}$ but neither $a_{k-1}, \ldots, a_{k+l-1}$ nor $a_{m+1}, \ldots, a_{k+l-1}$ is a dragon. Let's note that we may partition all the dragons into such chains of dragons, hence it suffices to prove that the arithmetic mean of all heads of dragons in a single chain is greater than 2017.

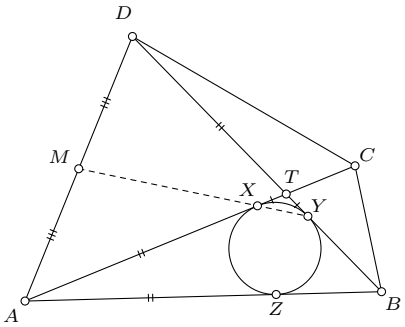
Suppose for a contradiction that there exists a chain of dragons such that the arithmetic mean of their heads is at most 2017. However, since $a_{m+1}, \ldots, a_{k+l-1}$ is not a dragon, it follows that the arithmetic mean of these numbers is also at most 2017. These two facts imply that the arithmetic mean of numbers $a_k, \ldots, a_{k+l-1}$ is at most 2017 contradicting it being a dragon. Henceforth, indeed it follows that the arithmetic mean of all the heads (counting multiplicities) is greater than 2017.

Geometry

Problem 1. Let $ABCD$ be a convex quadrilateral satisfying $AC = BD = AB$. Let M and N be the midpoints of the segments AD and BC respectively. Denote by T the common point of the diagonals. Prove that the line MN passes through the touching points of the incircle of $\triangle ATB$ with the sides AT and TB.

Solution:
Let X, Y and Z be the touching points of the incircle with AT, BT and AB respectively. Hence, $BY = BZ$ and therefore $DY = AZ = AX$.

Furthermore, $TX = TY$ so applying the Menelaus theorem to $\triangle ATD$ and the line XY, we obtain that this line passes through M. Similarly it passes through N. □



Problem 2. Equilateral triangle ABC is given. Let o be the circumcircle and ω be the incircle of ABC with common center O. Let P and Q be the points lying on the sides AC and AB respectively, such that O lies on PQ. Let γ_b and γ_c be the circles with diameters BP and CQ respectively. Show, that one point of intersection of γ_b and γ_c lies on o and second point lies on ω

Solution:

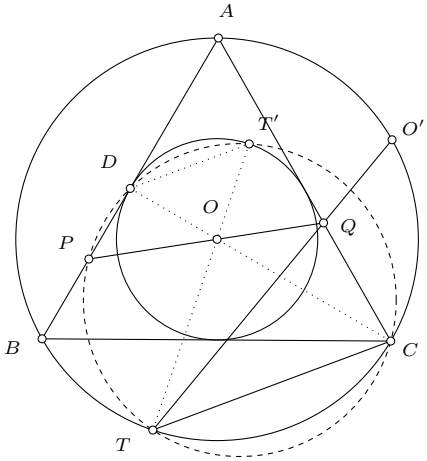
Let T be the anti-steiner point of line PQ with respect $\triangle ABC$. Denote by D the tangency point of the circle ω with side AB. Let T' be the point of intersection of half line $\overrightarrow{TO}$ with circle ω not lying on segment TO.

First of all, using anti-steiner point properties line TQ meets o at the point O' - reflection of the point O with respect side AC. It follows that $\sphericalangle QTC = 30°$. Analogously we prove that $\sphericalangle PTB = 30°$. Since $\sphericalangle BTC = 120°$, we obtain

$$\sphericalangle PTC = 90° = \sphericalangle BTQ$$

so circles γ_1 and γ_2 intersect at the point T.

On the other hand, by homothetic with center O transforming ω to o we have $DT' \parallel TC$. Since $TCT'D$ is isoscales trapezoid, the point T' lies on γ_1. Analogously we can prove that T' lies on γ_2.



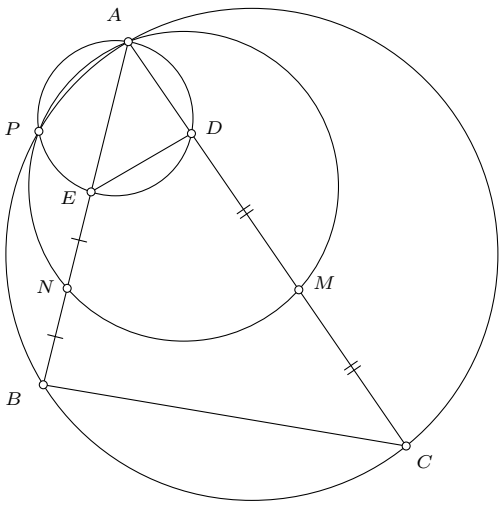
□

Problem 3. Let H be the ortocenter of $\triangle ABC$. The point P lies on circumcircle of $\triangle ABC$ and the point M is the midpoint of the side BC. Let T be the projection of H onto line AP. Prove that

$$MT = MP.$$

Solution:

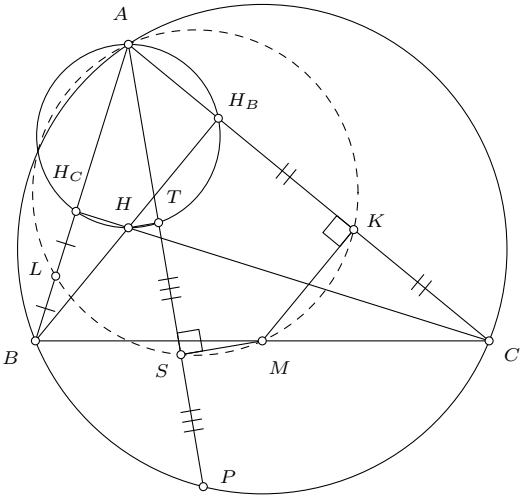
Lemma: The points D and E lie on sides AC and AB of $\triangle ABC$ respectively. The points M and N are the midpoints of CD and BE respectively. The circlumcircles of $\triangle ABC$, $\triangle AED$ and $\triangle AMN$ intersect at the one point.



Let P be the intersection point of the circumcircles of $\triangle AED$ and $\triangle ABC$. We see that $\sphericalangle PBA = \sphericalangle PCA$ and $\sphericalangle PEA = \sphericalangle PDA$ so $\triangle PBE \sim \triangle PCD$. Hence,

$$\triangle PBE \cup N \sim \triangle PCD \cup M$$

so $\sphericalangle PNA = \sphericalangle PMA$. It proves that the quadrilateral $APNM$ is cyclic.



Let H_B and H_C be the foots of altitudes from vertexes B and C respectively. Denote by K, L and S the midpoints of segments CH_B, BH_C and TP respectively.

As we know $\sphericalangle AH_CH = \sphericalangle ATH = \sphericalangle AH_BH = 90^\circ$ so the points A, H, H_B, H_C and T lie on circle with diameter AT. Using lemma pointed above the point K, L, S and A lie on the one circle. Moreover, $\sphericalangle MKA = \sphericalangle MLA = 90^\circ$ so the point M also lies on this circle. Therefore, $\sphericalangle MSA = 90^\circ$ implying $TM = MP$. □

Number Theory

Problem 1. Let N be a natural number. Does there exist an increasing arithmetic progression which has N terms and the sum of every term's digits is constant?

Solution:

For each n, denote by $s(n)$ the sum of digits of n. First, note that if $a, b \in \mathbb{N}$ such that $a + b = 10^m - 1$ then $s(a) + s(b) = s(10^m - 1) = 9m$. Let $t = 9(10^m - 1)$, where m is some fixed integer such that $\frac{10^m-1}{9} > N$. As $\text{nwd}(t, 10) = 1$, there exists a multiple of t, say kt, that is congruent to $10^m - 1$ in modulo 10^{2m}. Write $kt = a \cdot 10^{2m} + 10^m - 1$. Now for each $l = 0, 1, \ldots, \frac{10^m-1}{9}$, we have $(k+l)t = a \cdot 10^{2m} + (9l) \cdot 10^m + (10^m - 1 - 9l)$ so $s((k+1)t) = s(a) + s(9l) + s(10^m - 1 - 9l) = s(a) + 9m$ is constant for $l = 0, 1, \ldots, \frac{10^m-1}{9}$. Thus we have found an increasing arithmetic progression with at least N terms satisfying the required condition.

talkon's solution from AoPS

Problem 2. Prove that if $n\phi(n) = m\phi(m)$ then $m = n$

Solution:

We induct on the number of prime divisors of m. Clearly the result is true if m has no prime divisors, or $m = 1$.

First, if m has a larger prime divisor than all the prime divisors of n, then let that prime be p. Then $v_p(\phi(m^2)) > 0$ while $v_p(\phi(n^2)) = 0$, contradiction. We get a similar contradiction if the largest prime divisor of n is larger than all prime divisors of m. In other words, they share the same largest prime divisor. Let this be p; then $v_p(\phi(m^2)) = v_p(\phi(n^2))$. But this equals $2v_p(m) - 1 = 2v_p(n) - 1$, so $v_p(m) = v_p(n)$. But now we induct: if p^e is

the power of p dividing both m and n, then take $m' = m/p^e$ and $n' = n/p^e$. Since $x\phi(x)$ is multiplicative, we have $m'\phi(m') = n'\phi(n')$. By the inductive hypothesis it follows $m' = n'$, so $m = n$.

Casual_9's solution from AoPS

Problem 3. For an integer $x \geqslant 1$, let $p(x)$ be the least prime that does not divide x, and define $q(x)$ to be the product of all primes less than $p(x)$. In particular, $p(1) = 2$. For x having $p(x) = 2$, define $q(x) = 1$. Consider the sequence $x_0, x_1, x_2, \ldots$ defined by $x_0 = 1$ and

$$x_{n+1} = \frac{x_n p(x_n)}{q(x_n)}$$

for $n \geqslant 0$. Find all n such that $x_n = 1995$.

Solution:

Okay so note that $x_1 = 2$, $x_2 = 3$. Then $x_3 = \frac{3*2}{1} = 6 = 2*3$. We let this be 11. Then, $x_4 = \frac{3*2*5}{3*2*1} = 5$ let this be 100. I will prove in general the correspondence between binary numbers.

Assume via induction that this binary pattern holds. The base case is proved above. Now, assume it holds for the set x_{2^k} to $x_{2^{k+1}}$ and I prove that it works for $x_{2^{k+1}}$ to $x_{2^{k+2}}$.

Note that the numbers without the factor of the $k+1$th prime (this is the leading factor) are going to create the group 2^{k+1} to $3*2^k$ by induction hypothesis. Also, we get that for $3*2^k$ to 2^{k+2} , we get an added factor of the $k+1$th prime by all these numbers, therefore we are done by induction.

$1995 = 3*5*7*19$. Therefore, putting this in this form of induction, we get $(19)(17)(13)(11)(7)(5)(3)(2)_2$ (where we only include digits which are in the prime factorization). Therefore, we get 10001110_2. This is $2^7 + 2^3 + 2^2 + 2 = \boxed{142}$.

Binomial-theorem's solution from AoPS

Middle Match

Algebra

Problem 1. Show that if the integers $a_1; \ldots a_m$ are nonzero and for each $k = 0; 1; \ldots; n$ $(n < m-1)$, $a_1 + a_2 2^k + a_3 3^k + \ldots + a_m m^k = 0$; then the sequence $a_1, \ldots, a_m$ contains at least $n+1$ pairs of consecutive terms having opposite signs.

Solution:

Let $1 < k_1 < k_2 < \cdots < k_r \leqslant m$ be the indices where $\{a_k\}$ change signs. So $a_{k_j - 1} a_{k_j} < 0$, $\forall 1 \leqslant j \leqslant r$. Obviously $r \geqslant 1$. Suppose there are at most n such pairs, that is $r \leqslant n$.

Notice we have $\sum_{k=1}^{m} a_i f(i) = 0$, where f is any real-coefficient polynomial of degree no greater than n. Now we let $f(x) = \prod_{j=1}^{r} (x - k_j)$. We observe $a_i f(i) = 0, \forall i \in \{k_1, k_2, \cdots, k_r\}$ and other terms share signs with $a_1 f(1)$. Absurd!

xxp2000 - AoPS's user solution

Problem 2. Find all $a \in \mathbb{R}$ for which there exists a non-constant function $f : (0, 1] \to \mathbb{R}$ such that

$$a + f(x + y - xy) + f(x)f(y) \leqslant f(x) + f(y)$$

for all $x, y \in (0, 1]$.

Solution:

$x = y = 1$ gives $a \leqslant f(1) - f(1)^2 = \frac{1}{4} - (f(1) - \frac{1}{2})^2 \leqslant \frac{1}{4}$.

For $a < \frac{1}{4}$ we can take $f : (0, 1] \to [\frac{1}{2}, \min\{\frac{3}{4} - a, 1\}]$ arbitrary. Then

$$a + f(x + y - xy) + f(x)f(y) - (f(x) + f(y)) =$$
$$= a + f(x + y - xy) + (1 - f(x))(1 - f(y)) - 1 \leqslant a + (\frac{3}{4} - a) + \frac{1}{2} \cdot \frac{1}{2} - 1 = 0$$

for all $x, y \in (0, 1]$.

For $a = \frac{1}{4}$ we get $(f(1) - \frac{1}{2})^2 \leqslant 0$ and $f(1) = \frac{1}{2}$. Then $y = 1$ gives $f(x) \geqslant \frac{1}{2}$ for all $x \in (0, 1]$. $x = y$ gives

$$f(2x - x^2) \leqslant 2f(x) - f(x)^2 - \frac{1}{4} = \frac{3}{4} - (1 - f(x))^2 \leqslant \frac{3}{4}$$

for all $x \in (0, 1]$.

On the other hand, $2x - x^2 = 1 - (1 - x)^2$ runs through all of $(0, 1]$ for $x \in (0, 1]$. So $t = \sup\{f(x) : x \in (0, 1]\} \in [\frac{1}{2}, \frac{3}{4}]$ exists. But then

$$t = \sup_{x \in (0,1]} f(2x - x^2) \leqslant \sup_{x \in (0,1]} (\frac{3}{4} - (1 - f(x))^2) = \frac{3}{4} - (1 - t)^2$$

so $t = \frac{1}{2}$. It gives $f \equiv \frac{1}{2}$ constant. Contradiction!

Combinatorics

Problem 1. In an $m \times n$ rectangular grid, where m and n are odd integers, 1×2 dominoes are initially placed so as to exactly cover all but one of the 1×1 squares at one corner of the grid. It is permitted to slide a domino towards the empty square, thus exposing another square. Show that by a sequence of such moves, we can move the empty square to any corner of the rectangle.

Solution:

We will consider in our solution only those squares of the grid, whose both vertical and horizontal coordinates differ from those of the corner of the grid by an even amount. Note that these are the only squares, which we can make empty and, as we shall show, we can make any of them empty. Let's make the following graph on the set of vertices described above. Every such vertex, except for the corner, is covered by some domino piece. Note that since coordinates of every such vertex have the same parity as the corner, by reflecting this vertex about

the other square of covering domino, we obtain another point inside the grid, and what's more, also a vertex of our set. We connect these two vertices with an edge.

Let's notice further that no two vertices of this set can give rise to the same edge. Furthermore, this graph is cycle-free, since every cycle consisting of dominoes encloses an odd number of squares, which obviously cannot be covered by dominoes (let's recall that initially the only empty square in the grid is in the corner, hence it cannot be enclosed by a loop of dominoes). Therefore, we obtain a graph on a set of vertices which is cycle-free and whose number of edges is exactly one less than the number vertices. Therefore, it is a tree. To empty any chosen square from this set, we simply find a path in the tree and move the dominoes according to it. Therefore, since m and n are odd integers, hence other corners are also in our set and thus it is possible to find a sequence of moves that finally empties any of them.

Problem 2. Paweł and Ania eat a pizza of $2n$ pieces. Each piece contains a distinct amount of olives between 1 and $2n$. Paweł eats the first piece, and the two players alternately eat a piece neighbor of an eaten piece. However, neither Paweł nor Ania like olives, so they will choose pieces that minimizes the total amount of olives they eat. For each arrangement σ of the olives, let $s(\sigma)$ the minimal amount of olives that Paweł can eat, considering that both play in the best way possible. Let $S(n)$ the maximum of $s(\sigma)$, considering all arrangements.

(a) Prove that $n^2 - 1 + \lfloor \frac{n}{2} \rfloor \leq S(n) \leq n^2 + \lfloor \frac{n}{2} \rfloor$

(b) Prove that $S(n) = n^2 - 1 + \frac{n}{2}$ for each even n.

Solution:

Let $a_1, a_2, \ldots, a_{2n}$ be the amounts of olives on consecutive pieces of pizza (note that they depend on σ). Denote by $s_1(\sigma)$ and $s_2(\sigma)$ the sum of a_i's with i being odd and i being even, respectively. Since

$$s_1(\sigma) + s_2(\sigma) = 1 + 2 + \ldots + 2n = \frac{2n \cdot (2n+1)}{2} = 2n^2 + n$$

hence one of these numbers, without loss of generality let's assume it $s_1(\sigma)$ is not greater than $n^2 + \lfloor \frac{n}{2} \rfloor$. The strategy for Paweł to obtain exactly $s_1(\sigma)$ pieces is the following: Paweł takes the first piece of pizza (that is, the one with a_1 olives). Ania can only take either second or $2n$-th piece, hence a piece with even index. No matter what she chooses Paweł can take a piece with odd index again leaving Ania with the choice between two pieces with even indices. Proceeding so, Paweł will take all the pieces with odd indices at the end obtaining exactly $s_1(\sigma)$. This strategy assures us that Paweł can always take at most $n^2 + \lfloor \frac{n}{2} \rfloor$ olives, hence $S(n) \leq n^2 + \lfloor \frac{n}{2} \rfloor$.

For the second part of the inequality we only need to prove that there exists a permutation σ and Ania's strategy such that Paweł will take at least $n^2 - 1 + \lfloor \frac{n}{2} \rfloor$ olives. Let's take the following permutation

$$(a_1, a_2, \ldots, a_{2n}) = \begin{cases} (2, 1, 3, 4, 6, 5, 7, 8, \ldots, 2n-1, 2n) & \text{if } 2 \mid n \\ (2, 1, 3, 4, 6, 5, 7, 8, \ldots, 2n, 2n-1) & \text{if } 2 \nmid n \end{cases}$$

Note that using this permutation we obtain $\lfloor \frac{n}{2} \rfloor$ quadruples of form $4k+2, 4k+1, 4k+3, 4k+4$ ($k \in \mathbb{Z}$) of olives lying on four consecutive pieces of pizza. The Ania's strategy will be simply to choose the piece of pizza paired up with the one Paweł has just chosen, that is if Paweł chooses $2k$, Ania chooses $2k-1$, if Paweł chooses $2k-1$, Ania chooses $2k$. This way Paweł takes at least $1 + 3 + 5 + \ldots + 2n - 1 = n^2$ olives. However, let's not forget about our quadruples. Due to Ania's strategy the first move in every such quadruple is made by Paweł and except for the very first move he will take either $4k+2$ or $4k+4$ olives. This makes him take additional $\lfloor \frac{n}{2} \rfloor - 1$ olives (-1 is there since he can start in some quadruple in $4k+1$ or $4k+3$ avoiding this way one additional olive). Therefore, indeed

$$n^2 - 1 + \lfloor \frac{n}{2} \rfloor \leq S(n) \leq n^2 + \lfloor \frac{n}{2} \rfloor.$$

To obtain a result for even n's we need to adjust a bit our Paweł's initial strategy. Let's note that there is a piece of pizza that has fewer olives than its neighbour - indeed, otherwise the sequence $a_1, a_2, \ldots, a_{2n}, a_1$, would be strictly increasing or decreasing leading to a contradiction. If $s_1(\sigma)$ and $s_2(\sigma)$ for a given σ are distinct, then using usual Paweł's strategy he can take at most $n^2 - 1 + \lfloor \frac{n}{2} \rfloor$ olives. Otherwise, the strategy is to start at the piece of pizza, which has fewer olives than its neighbours - assume without loss of generality that this is the first piece - and after Ania's move switch to eating only pieces with even indices using the previous strategy. This way he will take strictly less than $n^2 + \frac{n}{2}$ olives, which is the amount he would have taken if Paweł and Ania had switched their amounts of olives from the first move. Henceforth, indeed for even n, it follows that $S(n) = n^2 - 1 + \frac{n}{2}$.

Problem 3. A piece at the origin of the coordinate plane is given. Gábor and Mateusz act as following. First, Gábor marks on a lattice point, on which the piece cannot be anymore put. Then Mateusz moves the piece from the point $(x,\ y)$ to the point $(x+1,\ y)$ or $(x,\ y+1)$, at most k times. He cannot move the piece to a marked point. If Gábor wins when Mateusz can't move any pieces, then find all possible integers k such that Gábor will win in a finite number of moves, regardless of how Mateusz moves the piece.
Solution:

We will prove that for any k Gábor can win a finite number of moves. Let's consider a line given by $x+y=2Mk$, where M is to be specified later. On this line there are exactly $4Mk+1$ points, which we will refer to from this point as $A_0, A_1, A_2, \ldots, A_{4Mk}$ Notice that it takes at least M moves for B to get to line $x+y=Mk$. Until that moment let Gábor just mark $A_{4k}, A_{8k}, \ldots, A_{4Mk}$.

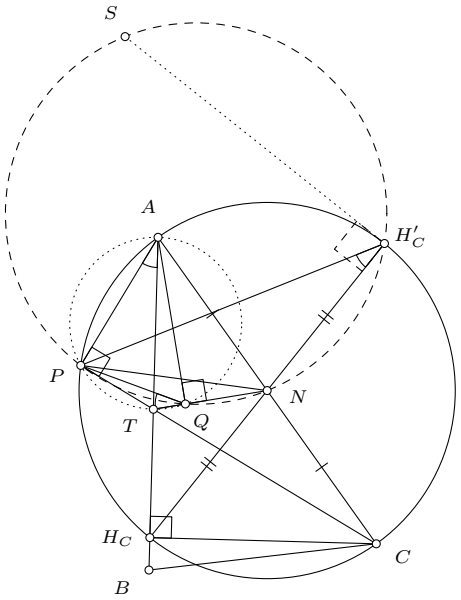
At that moment spectrum, that is the number of points to which Mateusz can get going only up or right, gets twice smaller. And again, Gábor may mark every $4k$-th point in the spectrum - this time the ones, which are congruent to 1 modulo $4k$. Proceeding in this way with $M=2^{4k+1}k$ allows A to mark all the current spectrum of B, that is, to get with this strategy through all the residues modulo $4k$ by the time he gets to $x+y=2^{4k+1}k$. If it so, then why do we need $M=2^{4k+1}k$ and not just 2^{4k}? Well, notice that at every step of our strategy we may not mark one of the points of the chosen residue, since the number of points on the line $x+y=2Mk$ equals to $4Mk+1$ and as shown in the first example, if we mark the points with index divisible by $4k$ we are one move short to mark all of them - in the example, we didn't mark point A_0. At every such move we can lose one such point leaving us at the line $x+y=Mk$ with at most $4k$ unmarked points in the spectrum. Due to the additional multiple of $2k$ in M we have still $4k$ moves to go, which we can use to mark the remaining points.

Just to finish the proof, it requires to handle the cases of Mateusz not going exactly k moves every move (as it was an unspoken rule above). To rule this out, if it takes strictly more moves for Mateusz to get from one marking line to other (i.e. the one on which Gábor switches residue according to which he are marking) Gábor does just random moves which doesn't really hurt him. The problem appears when Gábor gets almost to the line (for instance one line before) and then he can get between the lines in one move less than expected. To avoid this, we can start marking according to new residues when B is stricty nearer to the marking line than one move starting from marking points inside his spectrum so that no matter how he moves these points won't fall out of his spectrum when he passes through the marking line. That remark suffices to complete the proof.

Geometry

Problem 1. Given is an acute $\triangle ABC$. Choose T - an arbitrary point at the side AB. Let N be the midpoint of the segment AC. The foots of the perpendiculars from the point A to the segments TC and TN are the points P and Q respectively. Prove that center of the circumcircle of $\triangle PQN$ lies on the fixed line for all the points T from the side AB.

Solution:



Let S be the point on the circumcircle of $\triangle PQN$ such that SN is the diameter of this circle. By homothety we only need to prove that S lies on a fixed line.

Denote by H_C the projection of the point C onto the segment AB. Let the line H_CN meets circle with diameter AC again at the point H'_C. Since the quadrialteral $APTQ$ is cyclic, we get

$$\sphericalangle NH'_CP = \sphericalangle TAP = \sphericalangle PQT$$

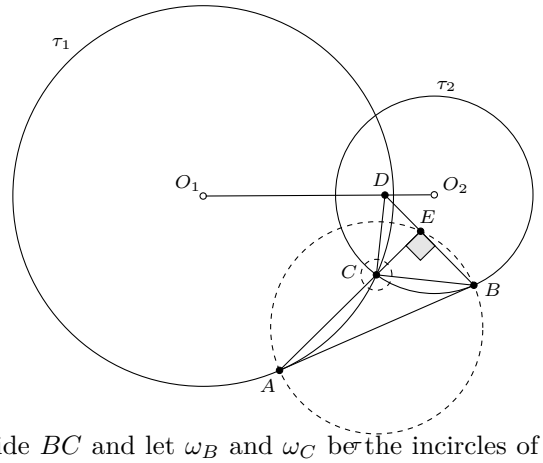
thus $PQNH'_C$ is cyclic.

It implies $\sphericalangle SH'_CN = 90^\circ$ so the line SH'_C is the tangent to the circle with diameter AC at the point H'_C. It shows that S lies on a fixed line.

□

Problem 2. Let τ_1, τ_2 be 2 intersecting circles with centers $O_1,\ O_2$ respectively. The point A lies on τ_1, the point B lies on τ_2 and AB is an ex-tangent line of τ_1 and τ_2. Suppose C be one of the intersection points of τ_1 and τ_2 nearer to AB. Let D be the point laying on O_1O_2 such that the line BD is perpendicular to AC. Prove that $\sphericalangle BCD = 90$.

Solution:
Let τ be the circle with diameter AB. Denote by $P(X, \omega)$ power of point X in respect to circle ω. It is easy to see that $P(O_1, \tau) = O_1B^2 = P(O_1, C)$ (we consider C as degenerated circle with zero radius). Analogously $P(O_2, \tau) = P(O_2, C)$ so $O_1,\ O_2$ is the radical axis of C and τ. As we know, $E = AC \cap DB$ lies on τ so $DC^2 = P(D, C) = P(D, \tau) = DE \cdot DB$. Equivalently: $\frac{DE}{DC} = \frac{DC}{DB}$ so $\triangle DCE \sim \triangle DBC$ which follows $\sphericalangle DCB = \sphericalangle DEC = 90^\circ$. □



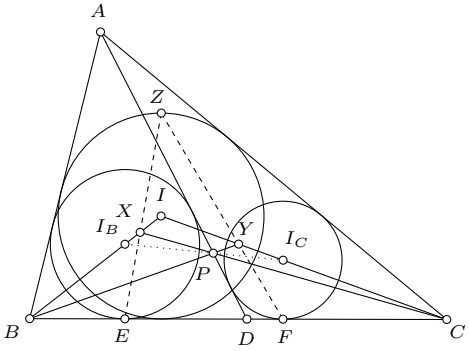
Problem 3. Let ABC be a triangle with incenter I. Let D be a point on side BC and let ω_B and ω_C be the incircles of $\triangle ABD$ and $\triangle ACD$ respectively. Suppose that ω_B and ω_C are tangent to segment BC at points E and F respectively. Let P be the intersection of segment AD with the line joining the centers of ω_B and ω_C. Let X be the intersection point of the lines BI and CP and let Y be the intersection point of the lines CI and BP. Prove that lines EX and FY meet on the incircle of $\triangle ABC$.

Solution:

Denote by ω incircle of $\triangle ABC$. Let Z be the point lying on ω such that the line tangent to ω at Z is parallel to BC and distinct from BC.

Note that point P is the insimilicenter of ω_B and ω_C. Moreover, the point C is the exsimilicenter of ω and ω_C. Thus by Monge theorem, the insimilicenter of ω_B and ω lies on line CP.

This insimilicenter should also lie on the line joining the centres of ω and ω_B, which is BI, hence it coincides with the point X. Therefore, $X \in EZ$ and analogously $Y \in ZF$ as desired.



□

Number Theory

Problem 1. Show that there exists a bijective function $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that for all $m, n \in \mathbb{N}_0$:

$$f(3mn + m + n) = 4f(m)f(n) + f(m) + f(n).$$

Solution:

define the function $G : 3\mathbb{N}_0 + 1 \to 4\mathbb{N}_0 + 1$ as:

$g(x) = 4f\left(\frac{x-1}{3}\right) + 1$

now note that f is bijective and so is g.

now let $m, n \in \mathbb{N}_0$,we have:

$$g\left((3m+1)(3n+1)\right) = g\left(3(3mn+m+n)+1\right) = 4f\left(3mn+m+n\right)+1 = \ = 4\left(4f(m)f(n)+f(m)+f(n)\right)+1 = (4f(m)$$

so for every $x, y \in 3\mathbb{N}_0 + 1$ we have:

$g(xy) = g(x) \cdot g(y)$

now notice that if we find a bijective function $g : 3\mathbb{N}_0+1 \to 4\mathbb{N}_0+1$ such that $g(xy) = g(x) \cdot g(y)$,then the function f can be found as $f(x) = \frac{g(3x+1)-1}{4}$. now let P_1, P_2 be the sets of prime numbers of the form $3k+1, 3k+2$ respectively and let Q_1, Q_2 be the sets of prime numbers of the form $4k+1, 4k+3$,now we define the bijective function h from $P_1 \cup P_2$ to $Q_1 \cup Q_2$ such that it pairs every element of P_1 with one element of Q_1,and the same thing for P_2, Q_2.

now we define g as:

$g(1) = 1$ and for every number $n \in 3\mathbb{N}_0 + 1$ greater than 1,let $n = \prod p_i$ (p_i's are prime numbers),then let:

$g(n) = \prod h(p_i)$

so obviously $g(xy) = g(x)g(y)$,now that we have found g,the function f can be easily obtained.

QED

Problem 2. Let A be a finite set of positive integers. Prove that there exists a finite set B of positive integers such that $A \subseteq B$ and

$$\prod_{x \in B} x = \sum_{x \in B} x^2.$$

Solution:

Suppose $\prod_{x\in A} x \leq \sum_{x\in A} x^2$. Let M be the maximal number of the set A. We can easily find $k > 0$ such that set $A' = A \cup \{M+1, M+2 \dots M+k\}$ satisfies $\prod_{x\in A'} x > \sum_{x\in A} x^2$.

Suppose $\prod_{x\in A} x > \sum_{x\in A} x^2$. Lets $A = \{x_1, x_2, ..., x_k\}$. For $n \geq k$ consider the sequence $x_{n+1} = x_1x_2...x_n - 1$ and denote $A_n = \{x_1, x_2, ..., x_n\}$. Moreover, let $s_n = \prod_{x\in A_n} x - \sum_{x\in A_n} x^2$ for $n \geq k$. From earlier assumption $s_k > 0$.

Using recursive definition of x_{n+1} we can calculate

$$s_{n+1} = \prod_{x\in A_{n+1}} x - \sum_{x\in A_{n+1}} x^2 = \prod_{x\in A_n} x - \sum_{x\in A_n} x^2 - 1 = s_n - 1.$$

It shows that taking $n = k + s_k$ we have $\prod_{x\in A_n} x - \sum_{x\in A_n} x^2 = s_n = 0$ and $A \subset A_n$ as desired.

Problem 3. Let m be a positive integer. Find all bounded sequences of integers $a_1, a_2, a_3, ...$ for which $a_n + a_{n+1} + a_{n+m} = 0$ for all $n \in \mathbb{N}$.

Solution:

Call a sequence satisfactory if it is periodic with period 3 and satisfies $a_1 + a_2 + a_3 = 0$. We claim the answer is all satisfactory sequences if $m \equiv 2 \pmod 3$ and the zero sequence otherwise. (Notice the zero sequence is satisfactory.)

If $m = 1$, then $a_n = 2a_{n+1}$ and $a_{n+1} = \frac{a_n}{2}$, which means that $a_n = \frac{a_1}{2^{n-1}}$ by induction. Thus a_1 is divisible by arbitrarily large powers of 2, which immediately implies $a_1 = 0$, so $a_n = 0$ for all n. Thus, suppose $m > 1$.

First, consider the characteristic equation $x^m + x + 1 = 0$ of the linear recurrence.

Lemma 1. It has no rational roots.

Proof. By the rational root theorem, any rational root must be a factor of ± 1, both of which fail to satisfy the equation (since odd numbers are never 0).

Lemma 2. It has no repeated roots.

Proof. Any repeated roots will also be a root of the derivative, $mx^{m-1} + 1$. Then $x^{m-1} = -1/m$. Substituting into the original equation, we get $(1 - 1/m)x = -1$, so x is rational, contradiction.

Lemma 3. If r is a root with $|r| = 1$, then $r^3 = 1$.

Proof. r^m and r must have negative imaginary parts, so since they both lie on the unit circle, we have $r^m = -r$ or $r^m = \overline{r}$. The first case fails since $r^m + r = -1$. Thus, r^m and r are conjugates, and since they sum to -1, their common real part is $-1/2$. Thus, they are cubic roots of unity, i.e. $r^3 = 1$.

Lemma 4. The primitive cubic roots of unity are roots of $x^m + x + 1 = 0$ if and only if $m \equiv 2 \pmod 3$.

Proof. Let r be a primitive cubic root of unity, satisfying $r^2 + r + 1 = 0$. Then $r^m + r + 1 = 0$ if and only if $r^m = r^2$, or $r^{m-2} = 1$, which happens if and only if $m \equiv 2 \pmod 3$.

By the general theory of linear recurrences, if $r_1, r_2, \ldots, r_m$ are the distinct roots of $x^m + x + 1 = 0$, then

$$a_n = \sum_{k=1}^{m} b_k r_k^n$$

for some constants $b_1, b_2, \ldots, b_m$. Because the sequence a_i is bounded, if $|r_i| > 1$, then $b_i = 0$. On the other hand, since a_i consists of integers, if $|r_i| < 1$, then b_i also equals 0. Hence, the only significant terms are when $|r_i| = 1$, which implies $r_i^3 = 1$. But then $a_{n+3} = a_n$, so a_n is periodic with period 3. Furthermore, $a_1 + a_2 + a_3 = 0$, so a_n is satisfactory. Thus, the solutions are the zero sequence when $m \neq 2 \pmod 3$ (since there will not be any terms with $|r_i| = 1$), and satisfactory sequences when $m \equiv 2 \pmod 3$.

Older Match

Algebra

Problem 1. Let $x,\ y$ and z be a positive real numbers. Show that

$$\frac{xy}{z}+\frac{yz}{x}+\frac{zx}{y}>2\sqrt[3]{x^3+y^3+z^3}.$$

Solution:

Set $a^2=\frac{yz}{x},\ b^2=\frac{zx}{y}$ and $c^2=\frac{xy}{z}$ where $a,\ b$ and c are positive numbers. The inequality can be rewritten as

$$a^2+b^2+c^2>2\sqrt[3]{a^3b^3+b^3c^3+c^3a^3}$$
$$\left(a^2+b^2+c^2\right)^3>8\sum_{cyc}a^3b^3$$
$$\sum_{cyc}a^6+3\sum_{sym}a^4b^2+6a^2b^2c^2>8\sum_{cyc}a^3b^3$$

Schur's inequality implies

$$\sum_{cyc}a^6+3a^2b^2c^2\geq\sum_{sym}a^4b^2$$

and Muirhead's inequality gives

$$\sum_{sym}a^4b^2\geq\sum_{sym}a^3b^3\geq2\sum_{cyc}a^3b^3.$$

Taking it together we obtain

$$\sum_{cyc}a^6+3\sum_{sym}a^4b^2+6a^2b^2c^2\geq4\sum_{sym}a^4b^2+3a^2b^2c^2\geq8\sum_{cyc}a^3b^3+3a^2b^2c^2>8\sum_{cyc}a^3b^3$$

so our proof is done. *bertram - AoPS's user solution*

Problem 2. Let $a_1,a_2...a_n$ be non-negative reals, not all zero. Show that that (a) The polynomial $p(x)=x^n-a_1x^{n-1}+...-a_{n-1}x-a_n$ has preceisely 1 positive real root R. (b) let $A=\sum_{i=1}^n a_i$ and $B=\sum_{i=1}^n ia_i$. Show that $A^A\leqslant R^B$.

Solution:

a)Consider the function $f(x)=\sum_{i=1}^n\frac{a_i}{x^i}$ and this function is decreasing for positive x .So there is exactly one R sucth that $f(R)=1$. b)Also by Jensen inequality we have $\sum_{i=1}^n\frac{a_i}{A}(ln\frac{A}{R^i})\leqslant ln(\sum_{i=1}^n\frac{a_i}{A}\frac{A}{R^i})=ln\ f(R)=0$

So $AlnA\leqslant BlnR$

Combinatorics

Problem 1. Subsets of the set $S=\{1,2,...,n\}$ are coloured witch 2017 colors. We call subsets A, B of S *good* if have the same color as $A\cap B$. Prove that if n is large enough, then good subsets exist.

Solution:

Consider a complete graph G with vertex $v_1,v_2,\ldots,v_{n+1}$ and for $1\leqslant i<j\leqslant n+1$ color the edge between v_i,v_j with color of $\{i,i+1,\ldots,j-1\}$. Now if $n>R_{2017}$ [7] there is a monochromatic triangle in G , namely $v_xv_yv_z$ for $x<y<z$. Then these three sets got same color : $A=\{x,x+1,\ldots,y-1\}$, $B=\{y,y+1,\ldots,z-1\}$, $A\cup B=\{x,x+1,\ldots,z-1\}$, and $A,\ B,C$ are good sets.

Problem 2. For a point P on the plane, denote by $\|P\|$ the distance to its nearest lattice point. Prove that there exists a real number $L>0$ satisfying the following condition:

For every $\ell>L$, there exists an equilateral triangle ABC with side-length ℓ and $\|A\|,\|B\|,\|C\|<10^{-2017}$.

Solution:

Define $\epsilon=10^{-2017}$. Let us take the vertices of the equilateral triangle to be $(0,0),(2a,2b),(a-\sqrt{3}b,\sqrt{3}a+b)$. Then it suffices to have

$$a^2+b^2=\frac{l^2}{4}:=x$$

[7] R_{2017} is $R_{2017}(3,3)$ Ramsey number.

and $||a||, ||b||, ||\sqrt{3}a||, ||\sqrt{3}b|| < \frac{\epsilon}{4}$. Let S denote the set of non-negative integers m with $||\sqrt{3}m|| < \frac{\epsilon}{12}$. We claim that for sufficiently large x, there exist $m, n \in S$ and $\alpha \in \left[0, \frac{\epsilon}{12}\right]$ such that $(m+\alpha)^2 + n^2 = x$. Once this is proved, we can set $a = m + \alpha$ and $b = n$ to obtain the desired equilateral triangle.

To prove the preceding claim, we need a lemma:

Lemma:
There exists a positive integer L, such that among every L consecutive non-negative integers, there is at least one that belongs to S.

This is a corollary of Dirichlet's approximation theorem, so we omit the proof.

Fix any sufficiently large x. Take m to be the largest integer in S not exceeding $\sqrt{x}$, then by Lemma $m \geqslant \sqrt{x} - L$ and so $x - m^2 \leqslant 2L\sqrt{x}$. Next take n to be the largest integer in S not exceeding $x - m^2$, then we can similarly deduce

$$0 \leqslant x - m^2 - n^2 \leqslant 2L\sqrt{2L\sqrt{x}} < 4L^2 x^{\frac{1}{4}}.$$

If α is such that $x = (m+\alpha)^2 + n^2$, then the leftmost inequality above gives $\alpha \geqslant 0$. Moreover, from $2m\alpha < 2m\alpha + \alpha^2 = x - m^2 - n^2 < 4L^2 x^{\frac{1}{4}}$ and $m > \sqrt{x} - L$, we can deduce $\alpha \leqslant \frac{\epsilon}{12}$. This completes the proof.

angiland - AoPS's user solution

Problem 3. A figure Φ composed of unit squares has the following property: if the squares of an $m \times n$ rectangle (m, n are fixed) are filled with numbers whose sum is positive, the figure Φ can be placed within the rectangle (possibly after being rotated) so that the sum of the covered numbers is also positive. Prove that a number of such figures can be put on the $m \times n$ rectangle so that each square is covered by the same number of figures.
Solution:

Let $\Phi_1, \ldots, \Phi_k$ be all possible ways of placing figure Φ in the rectangle. The statement of the problem can be reformulated in the following way: it is possible to attach to figures Φ_i such nonnegative widths d_i (d_i rational) that the sum of the widths of all figures Φ_i above every square of the rectangle equals to 1.

Let's assume for a contradiction that this is false. We will show that there exists an assignement of squares with numbers such that the sum of all numbers is positive and the sum of squares covered by the figure in any of the positions is nonpositive.

Let's introduce the following notation: indices j, $j = 1, \ldots, m \cdot n$, will denote the squares of the rectangle, indices i, $i = 1, \ldots, k$ - positions of the figure in the rectangle.

Let $P_{ij} = 1$ if j-th square is covered by figure Φ_i, $P_{ij} = 0$ otherwise. Then the choice of numbers $\{d_i\}$ corresponds to filling the squares of the rectangle with numbers $\theta_j = 1 - \sum_i d_i P_{ij}$, characterising the difference between this covering of the rectangle from the uniform one. According to our assumption, all numbers θ_j cannot be all 0.

Let's choose numbers $d_j \geq 0$ so that the amount of the difference $|\theta|$ is minimal, where $|\theta|^2 = \sum_j \theta_j^2$. We will prove that the obtained numbers θ_j yield a desired filling of rectangle with numbers.

Let's change one d_i into $d_i' = d_i + x$, $x \geq -d_i$. Then $\theta_j' = \theta_j - xP_{ij}$, hence

$$|\theta'|^2 = \sum_j \theta_j'^2 = \sum_j \theta_j^2 - 2x \sum_j P_{ij} + x^2 \sum_j P_{ij}^2, \quad \text{i.e.} \quad |\theta'|^2 = y(x) = ax^2 - 2b_i x + c$$

Here $a = \sum_j P_{ij}^2 = \sum_j P_{ij} = N$, where N is the number of square covered by N, $c = |\theta|^2$. Quadratic trinomial $y = y(x)$ described on the set $x \geq -d_i$ attains its minimum at $x = 0$ in case of $b_i = 0$ if $d_i > 0$ and in case of $b_i \leq 0$ if $d_i = 0$. This way, assuming that $|\theta|$ is minimal under the choice of $\{d_i\}$, we obtain that that if $d_i > 0$, then $d_i = \sum_j P_{ij} = 0$ and if $d_i = 0$, then $b_i \leq 0$. This means that the sum of the numbers covered by the figure Φ_i (which equals $\sum_j b_i P_{ij}$) is nonpositive. On the other hand, the sum of all numbers in the rectangle is positive,

since it is equal to $\sum_j \theta_j$ and $\sum_j \theta_j = \sum_j \theta_j^2$. Indeed,

$$\sum_j \theta_j^2 = \sum_j \theta_j\big(1 - \sum_i d_j P_{ij}\big) = \sum_j \big(\theta_j - \sum_i d_i(\theta_j P_{ij})\big) = \sum_j \theta_j - \sum_{i,d_i>0} d_i \sum_j \theta_j P_{ij} = \sum_j \theta_j$$

since $b_i = 0$ when $d_i > 0$.

Solution from

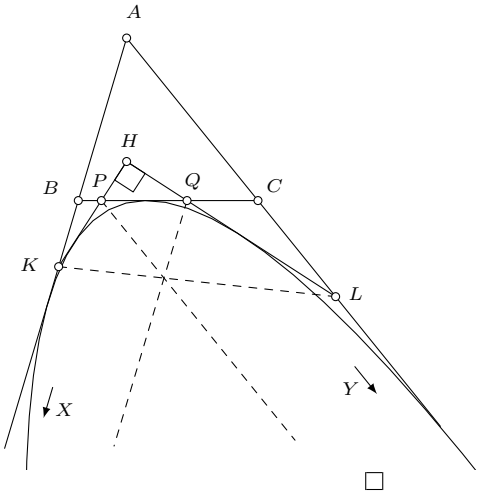
Geometry

Problem 1. Let H the orthocenter of $\triangle ABC$. The lines l_1 and l_2 pass through H and $l_1 \perp l_2$. The line l_1 intersects the lines AB and BC at the points K and P respectively. The line l_2 intersects the lines AC and BC at the points L and Q respectively. The line passing through the point Q and parallel to the line AB intersects the line passing through P and parallel to the line AC at the point T. Prove that the points $K,\ T$ and L are collinear.

Solution:

Let ζ be the parabola tangent to the lines $\{AB, AC, BC, HK\}$. From well-known lemma H lies on the directrix of ζ. We will use next well-known lemma i.e. if we draw line tangent to parabola passing through point lying on its directrix they will be perpendicular. Given by it the line HL is tangent to ζ.

Denote by X and Y intersection points of line at infinity with the lines AB and AC respectively. Notice that ζ is inscribed into hexagon $PKXYLQ$. Given by Brianchon's theorem lines $QX,\ PY$ and KL intersects at the one point - T.



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Problem 2. Let ABC be an acute triangle with the orthocenter H. On the segments HA, HB and HC, we choose three points A_1, B_1 and C_1 respectively such that

$$\sphericalangle BA_1C = \sphericalangle CB_1B = \sphericalangle AC_1B = 90^\circ.$$

Denote by $A_2 = BC_1 \cap CB_1, B_2 = CA_1 \cap AC_1$ and $C_2 = AB_1 \cap BA_1$.

a) Prove that the sides of the hexagon $A_1B_2C_1A_2B_1C_2$ are tangent to the one circle.

b) Let J be a incenter of the hexagon $A_1B_2C_1A_2B_1C_2$. Prove that the lines A_1A_2, B_1B_2, C_1C_2 and HJ are concurrent.

Solution:

First of all, lets introduce one lemma.

Lemma: Given $\mathcal{T}_1,\ \mathcal{T}_2,\ \mathcal{T}_2$ with common perspectrix l. Perspectors of any two triangles form the set $\{T_1,\ T_2,\ T_3\}$ are collinear.

Lets apply projective transformation which send common perspectrix into line at infinity. Now perspectors of our triangles are their centres of homothety. By D'Alembert theorem they are all collinear.

Lets come back to the main problem. Let the points $H_A,\ H_B$ and H_C be the feet of altitudes from vertexes $A,\ B$ and C in $\triangle ABC$ respectively. Denote by $X := B_2C_1 \cap BC$ and $Y := C_2B_1 \cap BC$.

By similarity of triangles we have

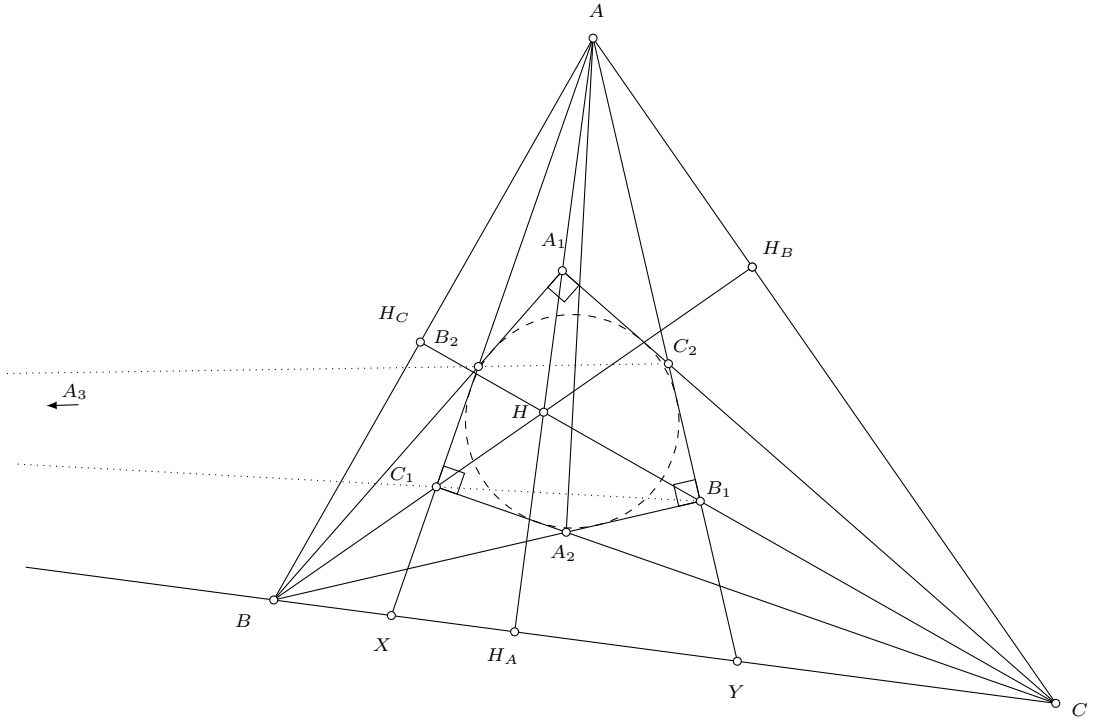
$$AB_1{}^2 = AH_C \cdot AC = AH_B \cdot AB = AC_1{}^2.$$

Moreover, $\sphericalangle AB_1A_2 = \sphericalangle AC_1A_2 = 90^\circ$ so the quadrilateral $AB_1A_2C_1$ is a kite. It implies that the quadrilateral $AB_1A_2C_1$ has incircle ω_A, define ω_B and ω_C similarly.

Assume that $\omega_A,\ \omega_B$ and ω_C are all distinct. The points A_1, B_1 and C_1 are exsimilicenter of $\omega_B \sim \omega_C, \omega_C \sim \omega_A$ and $\omega_A \sim \omega_B$ respectively. From D'Alembert theorem A_1, B_1, C_1 are collinear which is not true so at least two circles are the same. It implies, that $\omega_A = \omega_B = \omega_C$ so we are done with a) point.

Notice that $(A, X; C_1, B_2) = (A, H_A; H, A_1) = (A, Y; B_1, C_2)$ so lines BC, B_1C_1 and B_2C_2 are concurrent at the point A_3. Similarly, we can prove CA, C_1A_1 and C_2A_2 are concurrent at the point B_3 and AB, A_1B_1 and A_2B_2 are concurrent at the point C_3 .

Since $\triangle ABC$ and $\triangle A_1B_1C_1$ are perspective with perspector H, from Desargue theorem A_3, B_3 and C_3 are collinear at their perspectrix τ. Hence, $\triangle ABC,\ \triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ are perspective with common perspectrix τ given by our lemma perspector of any two triangles of $\{\triangle ABC, \triangle A_1B_1C_1, \triangle A_2B_2C_2\}$ are collinear. It proves that the lines A_1A_2, B_1B_2, C_1C_2 and HJ are concurrent.



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Problem 3. Given a $\triangle ABC$ has incenter at the point I. A line passes through I and intersects circumcircles of $\triangle IBC, \triangle ICA$ and $\triangle IAB$ again at the points $D,\ E$ and F respectively. Prove that the Euler lines of $\triangle DBC, \triangle ECA$ and $\triangle FAB$ are either concurrent or parallel.

Solution:

Let $\triangle I_AI_BI_C$ be the excentral triangle of $\triangle ABC$ and the point O be the circumcenter of $\triangle I_AI_BI_C$. Let the point O_D, O_E and O_F be the circumcenter of $\triangle BIC$, $\triangle CIA$, and $\triangle AIB$ respectively. Let l be the line perpendicular at the point I to $\overline{DEF}$. Let V be the anti-steiner point of l with respect to $\triangle I_AI_BI_C$ and let $\triangle V_AV_BV_C$ be the image of the medial triangle of $\triangle I_AI_BI_C$ under the homothety at point V and scale 2.

First of all, the line l passes through the antipode D' of the point D in the circumcircle of $\triangle I_ACB$. Moreover, by property of anti-steiner point, isogonal conjugate of line AV with respect to $\sphericalangle I_BI_AI_C$ is perpendicular to line l. Hence the lines I_AD' and I_AV are isogonal conjugated with respect to $\sphericalangle I_BI_AI_C$.

Denote by H_D the orthocenter of $\triangle DBC$. It is well known that D' is the reflection of the point H_B with respect to the midpoint of side BC. Since V lies on the circumcircle of $\triangle I_AI_BI_C$ we have

$$\triangle I_ABC \cup O_D \cup D' \cup H_D \sim \triangle I_AI_BI_C \cup O \cup V \cup V_A$$

Let S be the point such that $\triangle V_AV_BV_C \cup O \sim \triangle O_DO_EO_F \cup S$. As $\triangle V_AV_BV_C$ and $\triangle O_DO_EO_F$ have parallel sides, we have $V_AO \parallel O_DS$

From earlier conclusions we know that the lines O_DH_D and V_AO are anti-parallel in $\sphericalangle I_BI_AI_C$ so they are also anti-parallel in $\sphericalangle O_EO_DO_F$. Therefore, the line O_DH_D is the isogonal conjugate of the line O_DS with respect to $\sphericalangle O_EO_DO_F$.

Analogously we prove that isogonal conjugate of the point S with respect to $\triangle O_DO_EO_F$ lies on the Euler lines of $\triangle ECA$ and $\triangle FAB$

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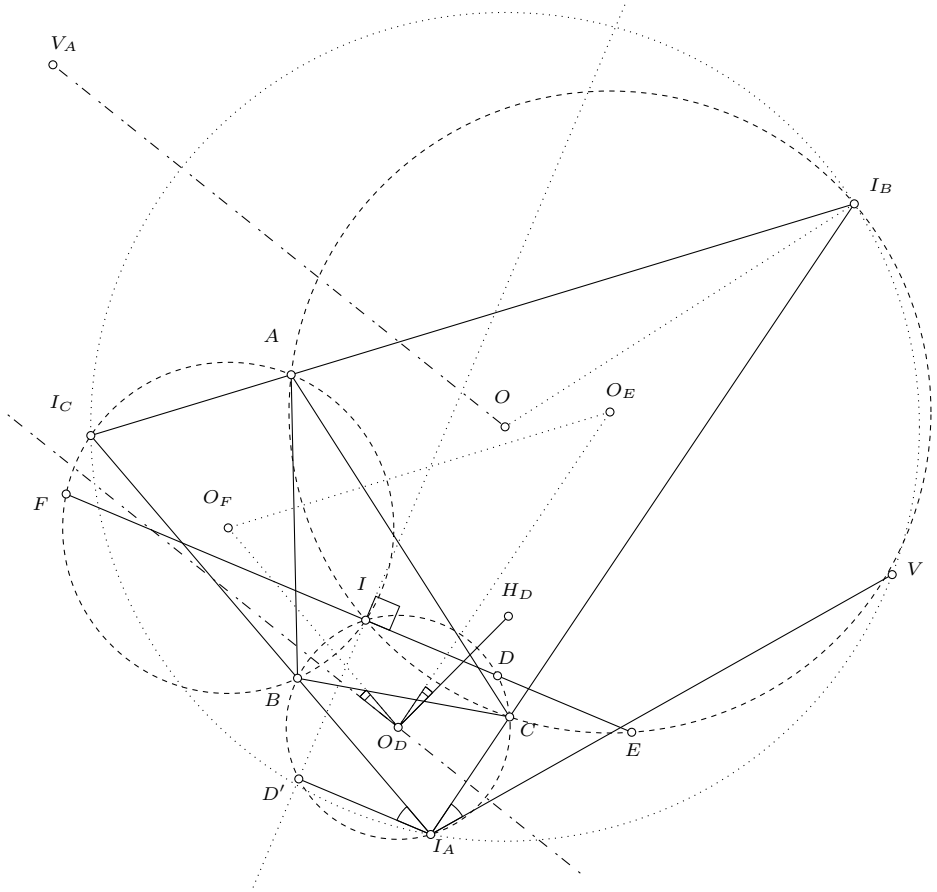

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Number Theory

Problem 1. Let $\varphi(n)$ denote the number of positive integers less than n that are relatively prime to n. Prove that there exists a positive integer m for which the equation $\varphi(n) = m$ has at least 2015 solutions in n.

Solution:

Lemma: For a prime p, we have $\phi(p(p-1)) = \phi((p-1)^2)$

Proof: Since phi is multiplicative, this is trivial.

Lemma 2: Let $S = [a_1, a_2, ..a_n]$ be a set and let $S' = [b_1, b_2, ..b_m]$ be a set. Suppose $\phi(a_1) = \phi(a_2) = .. = \phi(a_n)$ and $\phi(b_1) = \phi(b_2) = ... = \phi(b_m)$. Furthermore, suppose that for any i, j, if $d|\operatorname{nwd}(a_i, b_j)$, then d divides all the as and all the bs. Then $\phi(a_ib_j)$ is fixed for all i, j. This basically says we take the "product" of two sets, the phi's are still equal.

Proof: Just use the fact that ϕ is multiplicative

Now, we induct. Let $p_1 = 7$ and for any i, take $A_i = [(p_i - 1)^2, p_i(p_i - 1)]$, and define $A_1 = S_1$. Define S_i to consist of the productts of all numbers in S_{i-1} with all numbers in A_i. Define the p_i's so that $\operatorname{nwd}(p_i - 1, (p_1 -$

$1)(p_2-1)..(p_{i-1}-1))=1$ for all $i>1$ (this is easy with Dirichlet + CRT). Then by lemma 2, we can basically let $i>11$ to get $|S_i|>2015$ and we have over 2015 numbers with the same phi value.

Based on tastymath75025's solution on AoPs

Problem 2. Let p be the largest prime factor of $A=2^{2^n}+1$ $(n>3)$ Prove that $p\geq(n+2)2^{n+4}+1$

Solution:

Lemma: If $n\geqslant 2$, p is a prime, and $p|2^{2^n}+1$, then $2^{n+2}|p-1$. Proof: Let k be the order of 2 modulo p. Since $2^{2^n}\equiv -1 \pmod p$, $2^{2^{n+1}}\equiv 1 \pmod p$, so k must be a power of 2 less than or equal to 2^{n+1}. But if $k<2^{n+1}$, we may square the congreunce $2^k\equiv 1 \pmod p$ a total of $n-\log_2 k$ times to get $2^{2^n}\equiv 1 \pmod p$. But $2^{2^n}\equiv -1 \pmod p$ and $p\neq 2$, so we must have $k=2^{n+1}$. Because $n\geqslant 3$, $8\,|\,2^{n+1}\,|\,p-1$, whence $\left(\frac{2}{p}\right)=1$. Hence, $a^2\equiv 2 \pmod p$ for some integer a. The order of a must therefore be $2k$, so $2^{n+2}=2k\,|\,p-1$, as desired.

The problem is clearly true when $n=3$, so we may suppose that $n\geqslant 4$. Let $p_1\leqslant p_2\leqslant\ldots\leqslant p_m$ be primes such that $2^{2^n}+1=p_1p_2\ldots p_m$. By the lemma, there exist integers $a_1,a_2,\ldots,a_m$ such that $p_i=2^{n+2}a_i+1$ for each i. Hence, we have

$$2^{2^n}+1=(2^{n+2}a_1+1)(2^{n+2}a_2+1)\ldots(2^{n+2}a_m+1). \tag{1}$$

It follows that $2^{2^n}\geqslant(2^{n+2}+1)^m>2^{m(n+2)}$, whence

$$m<\frac{2^n}{n+2}.$$

Since $n\geqslant 4$, $2n+4\leqslant 2^n$. Taking (1) modulo 2^{2n+4} yields

$$\begin{aligned}1&\equiv 2^{2^n}+1\\&=(2^{n+2}a_1+1)(2^{n+2}a_2+1)\ldots(2^{n+2}a_m+1)\\&=2^{2n+4}(\text{some terms})+2^{n+2}(a_1+a_2+\ldots+a_m)+1\\&\equiv 2^{n+2}(a_1+a_2+\ldots+a_m)+1 \pmod{2^{2n+4}},\end{aligned}$$

whence

$$2^{n+2}|a_1+a_2+\ldots+a_m.$$

Since $a_m\geqslant a_{m-1}\geqslant\ldots\geqslant a_1$, we have $a_m\geqslant\frac{2^{n+2}}{m}$. Hence,

$$\begin{aligned}p_m&=2^{n+2}a_m+1\\&\geqslant 2^{n+2}\cdot\frac{2^{n+2}}{m}+1\\&>2^{n+2}\cdot\frac{2^{n+2}}{\frac{2^n}{n+2}}\\&=2^{n+4}(n+2),\end{aligned}$$

as desired.

Based on Zhero's solution on AoPS

Problem 3. Let p be an odd prime and n a positive integer. In the coordinate plane, seven distinct points with integer coordinates lie on a circle with diameter of length p^n. Prove that there exists a triangle with vertices at three of the given points such that the squares of its side lengths are integers divisible by p^{n+1}.

Solution:

Here's another proof for $m=7$ points (which I believe is optimal when $p\equiv 1 \pmod 4$). Let the m points be $(x_k,y_k)\in\mathbb{Z}^2$ $(1\leq k\leq m)$, $d_{j,k}$ be the distance between $(x_j,y_j),(x_k,y_k)$, and the center of the circle be (X,Y), so $(x_k-X)^2+(y_k-Y)^2=p^{2n}/4$ for every k. Since $m\geq 3$, X,Y must be rational. If $p\equiv 3 \pmod 4$, we must have $v_p(x_k-X),v_p(y_k-Y)\geq n$ for all k, so any three points will work (note that $2n\geq n+1$).

Otherwise, if $p\equiv 1 \pmod 4$, take integers a,b with $p=a^2+b^2$. From now on, we work in $\mathbb{Q}[i]$; since $\mathbb{Z}[i]$ is a UFD, we can define, for any fixed prime $q\in\mathbb{Z}[i]$, a q-adic valuation v_q on $\mathbb{Q}[i]$ satisfying $v_q(r/s)=v_q(r)-v_q(s)$ for any $r,s\in\mathbb{Z}[i]$ (where for $x\in\mathbb{Z}[i]$, $v_q(x)$ is the largest $t\geq 0$ with $q^t\mid x$)—observe that each element of $\mathbb{Q}[i]$ can be written in the form r/s for some $r\in\mathbb{Z}[i]$ and $s\in\mathbb{Z}$. Define $z_k=x_k+y_ki$, $Z=X+Yi$, and the

primes $\alpha = a + bi$, $\beta = \overline{\alpha} = a - bi$. Then $p = \alpha\beta$, $(z_k - Z)(\overline{z_k - Z}) = p^{2n}/4$, and $v_\alpha(z_k - Z) = v_\beta(\overline{z_k - Z})$ by conjugation (the same holds with α, β swapped), so

$$v_\alpha(z_k - Z) + v_\beta(z_k - Z) = v_\beta(\overline{z_k - Z}) + v_\beta(z_k - Z) = v_\beta(p^{2n}/4) = 2n$$

for each k.

Now note that if $v_\alpha(z_k - Z) < 0$ for some k, then $v_\alpha(z_j - Z) = v_\alpha((z_j - z_k) + (z_k - Z)) = v_\alpha(z_k - Z)$ for all $j \neq k$, since $z_j - z_k = (x_j - x_k) + (y_j - y_k)i \in \mathbb{Z}[i]$ implies $v_\alpha(z_j - z_k) \geq 0$. (Of course, the analogous statement for β also holds.) Hence by pigeonhole, there must exist 3 indices l_1, l_2, l_3 with

$$v_\alpha(z_{l_1} - Z) \leq v_\alpha(z_{l_2} - Z) \leq v_\alpha(z_{l_3} - Z) < v_\alpha(z_{l_1} - Z) + n.$$

(The "nontrivial case" is when $0 \leq v_\alpha(z_k - Z) = 2n - v_\beta(z_k - Z) \leq 2n$ for all k.) Then

$$v_\alpha(z_{l_j} - z_{l_k}) \geq \min(v_\alpha(z_{l_j} - Z), v_\alpha(z_{l_k} - Z)) \geq v_\alpha(z_{l_1} - Z)$$

and similarly,

$$v_\beta(z_{l_j} - z_{l_k}) \geq v_\beta(z_{l_3} - Z) = 2n - v_\alpha(z_{l_3} - Z) \geq n + 1 - v_\alpha(z_{l_1} - Z).$$

But $v_\beta(\overline{z_{l_j} - z_{l_k}}) = v_\alpha(z_{l_j} - z_{l_k})$ and $v_\alpha(\overline{z_{l_j} - z_{l_k}}) = v_\beta(z_{l_j} - z_{l_k})$ by conjugation, so

$$v_\alpha(d_{j,k}^2) = v_\alpha(z_{l_j} - z_{l_k}) + v_\alpha(\overline{z_{l_j} - z_{l_k}}) = v_\alpha(z_{l_j} - z_{l_k}) + v_\beta(z_{l_j} - z_{l_k}) \geq n + 1.$$

We similarly obtain $v_\beta(d_{j,k}^2) \geq n + 1$, so because $d_{j,k}^2 \in \mathbb{Z} \subset \mathbb{Z}[i]$, we must have $\alpha^{n+1}, \beta^{n+1} \mid d_{j,k}^2$. Thus $p^{n+1} = \alpha^{n+1}\beta^{n+1} \mid d_{j,k}^2$ (as α, β are distinct primes and therefore coprime), and we're done. (More precisely, the rational number $d_{j,k}^2/p^{n+1}$ must be an algebraic integer and thus an integer itself.)

Relays

Problem 1. Let N be the greatest integer multiple of 8, no two of whose digits are the same. What is the remainder when N is divided by 1000?
Ans: 120.

Problem 2. How many positive integers have exactly three proper divisors, each of which is less than 50?
Ans: 109.

Problem 3. Let S_i be the sum of the first i terms of the arithmetic sequence $3, 7, 11, 15, \ldots$. What are possible values of the expression

$$\frac{S_i}{i}(j-k)+\frac{S_j}{j}(k-i)+\frac{S_k}{k}(i-j)$$

subject to all possibilities of i, j, k
Ans: 0.

Problem 4. Let $M(n) = \{-1, -2, \ldots, -n\}$. For every non-empty subset of $M(2017)$ we consider the product of its elements. How big is the sum over all these products?
Ans: -1.

Problem 5. We call an isosceles trapezoid $PQRS$ interesting, if it is inscribed in the unit square $ABCD$ in such a way, that on every side of the square lies exactly one vertex of the trapezoid and that the lines connecting the midpoints of two adjacent sides of the trapezoid are parallel to the sides of the square.

Find the areas of all interesting trapzoids.
Ans: $\frac{1}{2}$.

Problem 6. Jane is 25 years old. Dick is older than Jane. In n years, where n is a positive integer, Dick's age and Jane's age will both be two-digit number and will have the property that Jane's age is obtained by interchanging the digits of Dick's age. Let d be Dick's present age. How many ordered pairs of positive integers (d, n) are possible?
Ans: 25.

Problem 7. Let $A_1, A_2, A_3, \ldots, A_{12}$ be the vertices of a regular dodecagon. How many distinct squares in the plane of the dodecagon have at least two vertices in the set $\{A_1, A_2, A_3, \ldots, A_{12}\}$?
Ans: 183.

Problem 8. The sequence a_n is defined by $a_1 = 1, a_2 = 2$ and every element is a sum of all the previous ones. What is the value of a_{12}?
Sol: 1536

Problem 9. Let $\star$ be an operation defined in the set of nonnegative integers with the following properties: for any nonnegative integers x and y,

(a) $(x+1)\star 0 = (0\star x)+1$
(b) $0\star(y+1) = (y\star 0)+1$
(c) $(x+1)\star(y+1) = (x\star y)+1$.

If $123\star 456 = 789$, find $246\star 135$
Ans: 579

Problem 10. Now run and find (Grzegorz?) and play with him the following game. At the beginning he will tell you a positive integer n and you choose whether you want to go first. Numbers $1, 2, \ldots, 10$ will be written on the paper. Alternately, you put $+$ or $-$ before the numbers. The first player wins if and only if the absolute value of the obtained sum doesn't exceed n.

Problem 11. How many 6-tuples (a, b, c, d, e, f) of natural numbers are there for which $a > b > c > d > e > f$ and $a + f = b + e = c + d = 22$ are simultaneously true?
Ans: 120

Problem 12. In decimal representation

34!=295232799039a041408476186096435b0000000.

Find the numbers a and b.
Ans: $(6, 2)$.

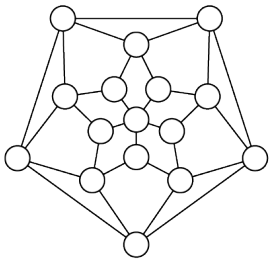
Problem 13. Let x_1, x_2, …, x_{10} be 10 numbers. Suppose that $x_i + 2x_{i+1} = 1$ for each i from 1 through 9. What is the value of $x_1 + 512x_{10}$?
Ans: 171.

Problem 14. We are given a biased coin with heads ending up with probability $\frac{2}{7}$ and tails otherwise. We toss it until we get heads. What is the expected number of tosses.
Ans: $\frac{7}{2} = 3,5$

Problem 15. In the interior of the square $ABCD$, the point P lies in such a way that $\sphericalangle DCP = \sphericalangle CAP = 25°$. Find the value of $\sphericalangle PBA$.
Ans: $40°$

Problem 16. A computer generates even integers half of the time and another computer generates even integers a third of the time. If a_i and b_i are the integers generated by the computers, respectively, at time i, what is the probability that $a_1b_1 + a_2b_2 + \cdots + a_6b_6$ is an even integer?
Ans: $\frac{364}{729} = \frac{3^6+1}{2\cdot 3^6} = \frac{1}{2} + \frac{1}{2\cdot 3^6}$

Problem 17. At each of the sixteen circles in the network below stands a student. A total of 3360 coins are distributed among the sixteen students. All at once, all students give away all their coins by passing an equal number of coins to each of their neighbors in the network. After the trade, all students have the same number of coins as they started with. Find the number of coins the student standing at the center circle had originally.



Ans: 280

Problem 18. Let $f(x) = \dfrac{9^x}{9^x + 3}$. Compute $\displaystyle\sum_k f\left(\frac{k}{2017}\right)$, where k goes over all integers k between 1 and 2017 which are coprime to 2017.
Ans: 1008

Problem 19. A function $f : \mathbb{Z} \to \mathbb{R}$ satisfies the equation:

$$f(n) = \begin{cases} n - 100 & \text{for } n > 2000 \\ f(f(n + 101)) & \text{for } n \leq 2000 \end{cases}$$

Find $f(1)$.
Ans: 1901.

Problem 20. Go play game with Grzegorz. You will be given two piles of (cookies or something like that) In each move you may eat one of the sweets from one pile, one from each of the piles or move one sweet from one pile to the another. Whoever empties the piles, wins.

Problem 21. Let S_1, S_2, ..., S_{125} be 125 sets of 5 numbers each, comprising 625 distinct numbers. Let m_i be the median of S_i. Let M be the median of m_1, m_2, ..., m_{125}. What is the greatest possible number of the 625 numbers that are less than M?
Ans: 436

Problem 22. Consider the following 50-term sums:
$S = \frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + ... + \frac{1}{99\cdot 100}$,
$T = \frac{1}{51\cdot 100} + \frac{1}{52\cdot 99} + ... + \frac{1}{99\cdot 52} + \frac{1}{100\cdot 51}$.
Express $\frac{S}{T}$ as an irreducible fraction.
Ans: $\frac{151}{2}$

Problem 23. Define a sequence $u_n, n = 0, 1, 2, 3...$ as follows: $u_0 = 0$, $u_1 = 1$ and, for each $n \geqslant 1$, u_{n+1} is the smallest positive integer such that $u_{n+1} > u_n$ and $u_1, u_2, ..., u_n, u_{n+1}$ contains no three elements that are in arithmetic progression. Find u_{100}.
Ans: 981

Problem 24. Determine all positive integers n such that the number $n(n + 2)(n + 4)$ has at most 15 positive divisors.
Ans: $1, 2, 3, 4, 5, 7, 9$.

Problem 25. The bisectors of the angles A and B in triangle ABC meet the sides BC and CA at the points D and E, respectively. Assuming that the angle A equals 45° and $AE + BD = AB$, determine the angle B.
Ans: 75°.

Problem 26. Consider a strip of 12 fieds, numbered from left to right with the integers 1 to n in ascending order. Each of the fields is colored with one of the colors 1, 2 or 3. Even-numbered fields can be colored with any color. Odd-numbered fields are only allowed to be colored with the odd colors 1 and 3. How many such colorings are there such that any two neighboring fields have different colors?
Ans: $4 \cdot 3^5$.

Problem 27. How many ways can the numbers $1, 2, 3, 4, 5$ be placed in a circular table with 6 seats, such that no two consecutive numbers are placed together? Any two arrangements obtained by rotation or reflection are considered to be the same.
Ans: 7.

Problem 28. Find all prime numbers p such that $2^p + p^2$ is also a prime number.
Ans: 3.

Problem 29. Let $f(n)$ denote an integer which is the closest to number $\sqrt{n}$. Compute

$$\frac{1}{f(1)} + \frac{1}{f(2)} + \ldots + \frac{1}{f(10000)}$$

Ans: 199.

Problem 30. Now run and find Grzegorz and play with him the following game. You are given number $N = 2310$. You alternately write composite divisors of N, but every time you choose one you must make sure that all the written divisors are not coprime and also there are no two divisors such that one divides the other. You decide who goes first. But remember you have only one chance.

Problem 31. Twenty-seven balls labelled from 1 to 27 are distributed in three bowls: red, blue, and yellow. What is the average of the possible values of the number of balls in the red bowl if the average labels in the red, blue and yellow bowl are 15, 3, and 18, respectively?
Ans: 16.

Problem 32. If $d(n)$ denotes the number of positive divisors of n, then evaluate

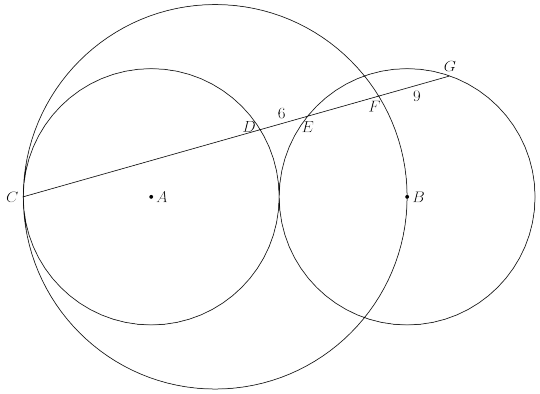
$$d(1) + d(2) + \ldots + d(24)$$

Ans: 84.

Problem 33. We define the weight W of a positive integer as follows: $W(1) = 0$, $W(2) = 1$, $W(p) = 1 + W(p + 1)$ for every odd prime p, $W(c) = 1 + W(d)$ for every composite c, where d is the greatest proper factor of c. Compute the greatest possible weight of a positive integer less than 100.
Ans: 12.

Problem 34. We say that some positive integer m *covers* the number 217 if $2, 1, 7$ appear in this order as digits of m (For instance, 217 is covered by **21**33267**8**, but not by 215993698). Let $k(n)$ be the number of positive integers that cover 217 and have exactly n digits, all different from 0. What is the remainder of $k(40)$ in division by 72?
Ans: 33.

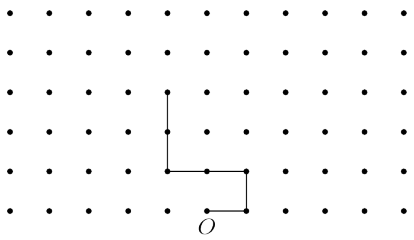
Problem 35. In the diagram below, the circle with center A is congruent to and tangent to the circle with center B. A third circle is tangent to the circle with center A at point C and passes through point B. Points C, A, and B are collinear. The line segment $\overline{CDEFG}$ intersects the circles at the indicated points. Suppose that $DE = 6$ and $FG = 9$. Find AG.



Ans: $9\sqrt{19}$.

Problem 36. Alice, Beth, Carla, Dana, and Eden play a game in which each of them simultaneously closes her eyes and randomly chooses two of the others to point at (one with each hand). A participant loses if she points at someone who points back at her; otherwise, she wins. Find the probability that all five girls win.
Ans: $\frac{24}{6^5} = \frac{1}{324}$.

Problem 37. The following figure shows a walk of length 6:



This walk has three interesting properties:

It starts at the origin, labelled O. Each step is 1 unit north, east, or west. There are no south steps. The walk never comes back to a point it has been to.

Let's call a walk with these three properties a northern walk. There are 3 northern walks of length 1 and 7 northern walks of length 2. How many northern walks of length 6 are there?
Ans: 239.

Problem 38. An integer $n \geqslant 3$ is called special if it does not divide $(n-1)!\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\right)$. Find all special numbers n such that $10 \leqslant n \leqslant 50$.
Ans: 7.

Problem 39. The sequence a_0, a_1, a_2, ... satisfies the recurrence equation

$$a_n = 2a_{n-1} - 2a_{n-2} + a_{n-3}$$

for every integer $n \geq 3$. If $a_{20} = 1$, $a_{25} = 10$, and $a_{30} = 100$, what is the value of a_{1331}?
Ans: 1.

Problem 40. You will be given a circle divided by 11 points into equal arcs. You and Grzegorz (you choose who goes first) alternately choose points to remove. The one after whose move all triangles are obtuse wins.